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Technical note

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The Performance of Smile-Implied Delta Hedging

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1 Introduction

Delta hedging is an important subject in financial engineering and more specifically risk management. By definition, delta hedging is the practice of neutralizing risks of adverse movements in an option's underlying stock price. Multiple types of risks arise on financial markets and the need for hedging is vast in order to reduce potential losses. Delta hedging is one of the most popular hedging methods for option sellers. So far, there has been a lot of theoretical work done on the subject. However, there has been very little discussion and research involving real market data. Hence, this subject has been chosen to shed some light on the practical aspect of delta hedging. More specifically, we focus on smile-implied delta hedging since it represents more accurately what actually happens on financial markets as opposed to the Black-Scholes model that assumes the option's volatility to be constant. In theory, delta hedging implies perfect hedging if the volatility associated with the option is non stochastic, there are no transaction costs and the hedging is done continuously. However, in practice, all three are unrealistic assumptions.

We contrast the performance of model-based delta hedging with the model-free smile-implied delta hedging introduced by Bates (2005). Therefore, in this situation, the strike price and market price of options are sufficient in order to determine our deltas and model risk is avoided. Based on an empirical study of options on the S&P500 index, we show that the smile-implied model presented by Bates (2005) yields the best results in terms of hedging errors when compared to a common benchmark delta hedging strategy. Our findings complement the preliminary evidence reported by Alexander and Nogueira (2007). On a sample of six months of options data, they find little difference in the hedging performance (measured by the aggregate P&L) between the smile-implied method and other benchmark models.

This technical note is organized as follows. The next section presents the data and its origin as well as all modifications made to it. In the third section, we explain in detail the methods used. The fourth section exposes the various results obtained. And finally, the last section gives the conclusions related to this work.

2 Data and definitions

For our analysis, we have used European S&P 500 index options. We have extracted the data from *OptionMetrics* and it ranges from September 3rd 2013 to August 29th 2014.
We define the initial moneyness of a contract as being the following ratio:

$$Initial\ moneyness = \frac{K}{S_0}$$

where K is the strike price of the contract and S_0 is the adjusted index price at the inception of the contract. It is assumed that out-of-the-money (OTM) calls and in-the-money (ITM) puts have an initial moneyness level exceeding 1.05; at-the-money (ATM) options have an initial moneyness level between 0.80 and 1.05 included; and that ITM calls and OTM puts have an initial moneyness level lower than 0.80. Furthermore, short-term options contracts are options that live for a total of 90 days or less and long-term options are options that live for a total of more than 90 days.

Table 1 reports the number of contracts used across moneyness and time to maturity.

Option Contracts (Call/Puts)		
	Short-Term	Long-Term
$\frac{K}{S_0} < 0.80$	439/439	380/380
$0.80 \leq \frac{K}{S_0} \leq 1.05$	550/550	489/489
$\frac{K}{S_0} > 1.05$	221/221	211/211

Table 1: Number of calls/puts on the S&P 500 in our data sample. $\frac{K}{S_0}$ represents the initial moneyness of the contract. Short-term options contracts are options that live for a total of 90 days or less and long-term options are options that live for a total of more than 90 days.

Figure 1 shows the evolution of the index.

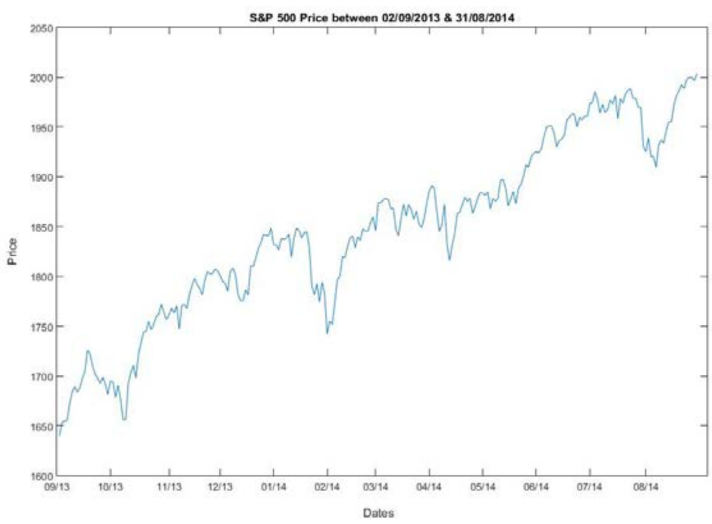

Figure 1: Evolution of the adjusted S&P500 index between September 1st 2013 and August 31st 2014.

It is important to note that the index price exhibits an upward trend. This observation will be useful to us when studying the deltas obtained.

In this paper we will use also the notion of implied volatility, the quantity obtained whenever the Black-Scholes model is inverted by using the option prices observed on the market. Furthermore, for an index option, volatilities exhibit a smile shaped curve when plotted which turned into a smirk after the 1987 crash. Generally, the implied volatilities of at-the-money options are distinctly lower than the implied volatilities in-the-money and out-of-the-money options.

3 Methodology

3.1 Delta hedging strategy

By definition, delta hedging an option is the practice of constructing a self-financing portfolio comprised of the underlying asset and a position in the money market account. The objective is to replicate as closely as possible the profits and losses of the derivative we have shorted. The difference in every delta-hedging strategy lies within the model used in order to compute the delta.

On the first day, the option is sold, an amount of delta is invested in the underlying asset and the difference is placed in the money market account. If the difference is negative, the amount is borrowed whereas if the difference is positive, the amount is lent. Every day, the underlying asset and option price fluctuate and consequently so does the delta and our different positions.

Let V_i represent the value of our hedging portfolio at the end of day i, M_i represent the money market account, g_i represent the premium of the option we want to hedge and Δ_i represent the number of shares of the underlying asset held between day i and day $i+1$.

The delta is calculated in the following way for any option contract *g:*

$$\Delta_g = \frac{\partial g}{\partial S}$$

Also, we define the daily compounding factor until the next rebalancing date to be:

- $B_i = e^{\frac{r_i}{365}}$ for Mondays to Thursdays
- $B_i = e^{\frac{3r_i}{365}}$ for Fridays (the money market account appreciates on weekend days)

where r_i is the overnight rate for day i.
On the first day, we have:

$$V_0 = \Delta_0 S_0 + M_0$$
$$M_0 = g_0 - \Delta_0 S_0$$

such that, the following day, in order for the portfolio to remain delta neutral and self-financed, the portfolio allocations are rebalanced.

We recalculate the new amount of underlying asset that is needed in the portfolio with the following day's delta and index price. If the amount that is required in the underlying asset increases, we withdraw money from the money market account in order to purchase the supplementary amount needed of the underlying asset. On the other hand, if the amount that is required in the underlying asset decreases, we sell the surplus and place the proceeds in the money market account.

Hence, the difference between the amount invested in the underlying asset between the end of the first day and the following day is:

$$difference = \Delta_0 S_1 - \Delta_1 S_1$$

Such that, on the following day, after rebalancing our positions, the money market account becomes:

$$M_1 = M_0 B_0 - difference = M_0 B_0 + (\Delta_1 - \Delta_0) S_1$$

Which is the sum of the gains/losses from the position in the underlying asset and the interest earned from the money market account.

Hence, we have:

- Before we rebalance our positions: $V_1^- = \Delta_0 S_1 + M_0 B_0$
- After we rebalance our positions: $V_1^+ = \Delta_1 S_1 + M_1$

The portfolio replicates the option in question and hence the absolute value of the difference between the replicating portfolio and the value of the option represents the hedging error.

Hence for day 1:

$$error_1 = |g_1 - V_1^-| = |g_1 - (\Delta_0 S_1 + M_0 B_0)|$$

This whole procedure is repeated daily during the lifetime of the option.

And, for each day i:

$$error_i = |g_i - (\Delta_{i-1} S_i + M_{i-1} B_{i-1})|$$

Finally, it is important to verify the self-financing condition every day:

$$\Delta_{i-1} S_i + M_{i-1} B_{i-1} = \Delta_i S_i + M_i$$

As an example, the first graph in figure 2 illustrates the evolution of the hedging error through time for a call option. The second graph in figure 2 compares the evolution of the real value of the option and its replicated value. We obtain similar graphs for put options.

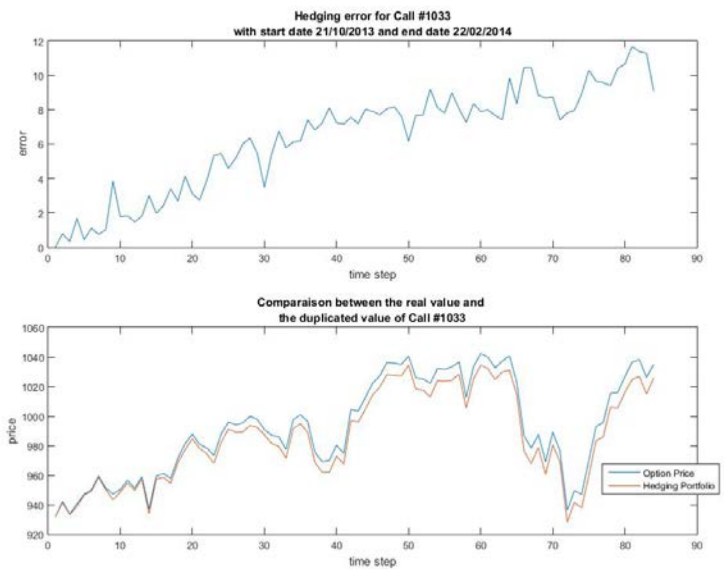

Figure 2: Delta hedging for a specific call on the S&P500 (Contract #1033).

3.2 Benchmark delta

Under the Black-Scholes model, the distribution of asset prices is assumed to be lognormal. For a European call option with underlying S, strike price K, dividend yield y, interest rate r, volatility σ, and time to maturity τ, its value under the Black-Scholes model is:

$$c(S,\tau,K) = S_t e^{-y(t,\tau)\tau} N(d_1) - K e^{-r(t,\tau)\tau} N(d_2)$$

where

$$d_1 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r(t,\tau) - y(t,\tau) + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$$\tau = T - t$$

$$\Delta_c = N(d_1)\, e^{-y(t,\tau)\tau}$$

$$\Delta_p = -N(-d_1) e^{-y(t,\tau)\tau}$$

$N(.)$ represents the standard normal cumulative distribution function.

To account for the observed smile, the Black-Scholes implied volatilities are smoothed through their respective strike price and time to maturity by means of a regression. This specification (commonly referred to as the "practitioners Black Scholes" model) is studied extensively in Dumas, Fleming and Whaley (1998).

Numerically, the following regression is applied every day to the Black-Scholes implied volatilities and available data. Specifically, for day j between September 1st 2013 and August 31st 2014, we have:

$$\sigma_j(T,K) = \max(0.01, \alpha_j + \beta_{1j}K + \beta_{2j}K^2 + \beta_{3j}T + \beta_{4j}T^2 + \beta_{5j}TK + \varepsilon_j)$$

where a minimum value is imposed in order to avoid negative volatilities.

3.3 Smile-implied delta

We now introduce a model-free method for computing our deltas. It avoids all shortcomings involved with the reliance on model assumptions.

The first result used in order to derive this method is that a homogeneous function of degree k is a function such that if one of its arguments is multiplied by a certain scalar t, then the output of the function is multiplied by t^k.

As initially discussed in Merton (1973), the value of a plain vanilla option is homogeneous of degree 1 in the underlying asset price under very mild assumptions. The homogeneity property entails that the price process of the underlying asset is scale-invariant, that is, the marginal distribution of its returns does not depend on its current price level. Put in simple words, should the underlying asset price be measured in dollars or in cents, the relative value of the option would be the same.

It has been shown by Bates (2005), by means of Euler's theorem, that if an option price g is homogeneous of degree one in the strike price K and underlying asset price S then the delta of an option can be computed in the following way:

$$g = S\frac{\partial g}{\partial S} + K\frac{\partial g}{\partial K}$$

which can be rewritten as:

$$\Delta_g = \frac{\partial g}{\partial S} = \frac{1}{S}\left(g - K\frac{\partial g}{\partial K}\right)$$

If computed this way, this delta is considered to be model-free since it involves no particular model and only relies on the data observed on the market. By contrast, the accurate determination of the smile-implied delta critically depends on (i) the quality of market information (i.e. reliable, liquid option prices), and (ii) the clean measurement of the partial derivative $\frac{\partial g}{\partial K}$.

Figures 3 and 4 show examples of observed option premiums on the market.

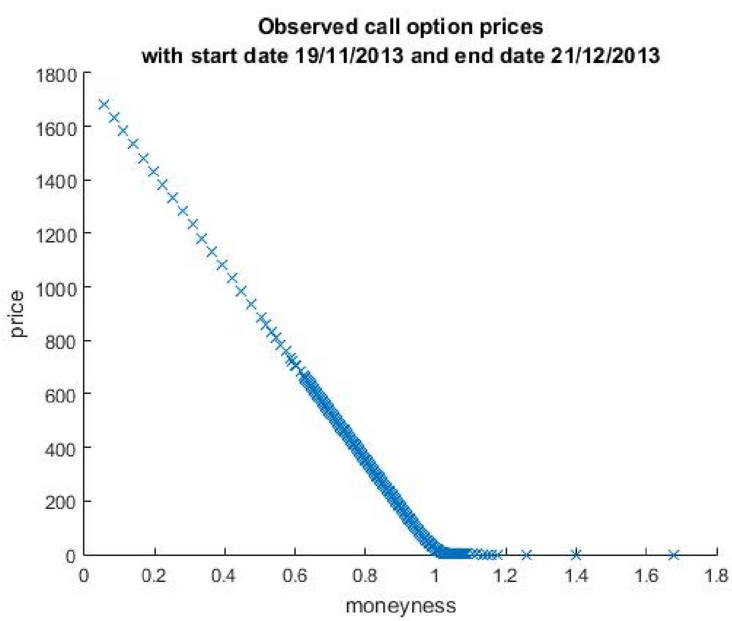

Figure 3: Observed premiums for a specific call option on the S&P500 index.

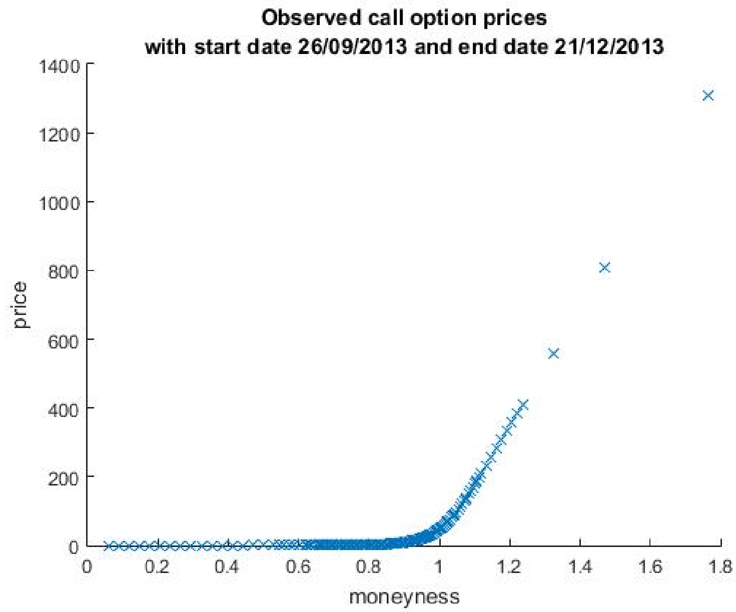

Figure 4: Observed premiums for a specific put option on the S&P500 index.

On a given day i, for an option g, we can use the central difference:

$$\frac{\partial g_i}{\partial K} \approx \frac{g_i(S, K + \Delta K) - g_i(S, K - \Delta K)}{2\Delta K}$$

For the first and last strike prices in our data, K_{min} and K_{max} respectively, we can use:

$$\frac{\partial g_i}{\partial K_{min}} \approx \frac{g_i(S, K_{min} + \Delta K) - g_i(S, K_{min})}{\Delta K}$$

$$\frac{\partial g_i}{\partial K_{max}} \approx \frac{g_i(S, K_{max}) - g_i(S, K_{max} - \Delta K)}{\Delta K}$$

We compute the delta for each day and use all available market premiums associated with a specific maturity. It is important to note that in practice, with raw data, ΔK is not fix. It varies depending on the available data on the market.

This method permits us to infer the delta of an option the way it is perceived by the market, however this does not mean the market it always accurate. Options are often mispriced because of different issues on the market such as liquidity and thus leading to overvalued or undervalued deltas.

To improve the calculation of $\frac{\partial g}{\partial K}$, we have decided to interpolate option premiums between each available point on the market.

Linear interpolation takes the following form for an option with a strike price K in $[K_i, K_{i+1}]$ and its associated premium g_K:

$$g_K = g_{K_i} + \left(g_{K_{i+1}} - g_{K_i} \right) \frac{K - K_i}{K_{i+1} - K_i}$$

We also considered cubic spline interpolation. Given that we have n available option premiums on the market, cubic spline interpolation takes the following form:

$$g(K) = a_i + b_i(K - K_i) + c_i(K - K_i)^2 + d_i(K - K_i)^3 \quad for\ K \in [K_i, K_{i+1}]$$
$$where\ i = 1, \dots, n-1$$

Furthermore, in our case, we use a natural cubic spline in order to smooth our option premiums. Meaning that, as boundary conditions, the second derivative of each polynomial is set to zero at the endpoints. That is:

$$g''(K_1) = g''(K_n) = 0$$

Other than the boundary conditions that we impose, we assume the following in order to obtain the coefficients of the above cubic spline:

- The interpolation with cubic splines must pass through all the available data on the market. Therefore, we have:
  - $g_i = a_i$ for $i = 1, \dots, n-1$
  - $g_n = a_{n-1} + b_{n-1}(K_n - K_{n-1}) + c_{n-1}(K_n - K_{n-1})^2 + d_{n-1}(K_n - K_{n-1})^3$
- Our cubic spline is continuous, meaning that when evaluated at the same data point, the splines on subsequent intervals are equal. Therefore, we have :
  - $g_i = a_i + b_i(K_{i+1} - K_i) + c_i(K_{i+1} - K_i)^2 + d_i(K_{i+1} - K_i)^3$ for $i = 1, \dots, n-2$
- Our cubic spline is differentiable at each point, meaning that its first derivative is continuous, therefore:
  - $g'(K_{i+1}) = b_i + 2c_i(K_{i+1} - K_i) + 3d_i(K_{i+1} - K_i)^2 = b_{i+1}$ for $i = 1, \dots, n-2$
  - $b_n = b_{n-1} + 2c_{n-1}(K_n - K_{n-1}) + 3d_{n-1}(K_n - K_{n-1})^2$
- The first derivative is differentiable as well, therefore:
  - $g''(K_{i+1}) = c_i + 3d_i(K_{i+1} - K_i) = c_{i+1}$ for $i = 1, \dots, n-2$

After interpolating our data (linearly or by means of cubic splines), we can approximate the partial derivative $\frac{\partial g}{\partial K}$ with any arbitrarily small ΔK.

Several values were tested but the hedging performance turned out to be little affected by the choice of ΔK provided it is sufficiently small. We report our results with ΔK = 1 (recall that S&P500 index option prices are quoted for every ΔK = 5).

For illustration purposes, Figure 5 presents three different plots of option prices for a given day and given maturity.

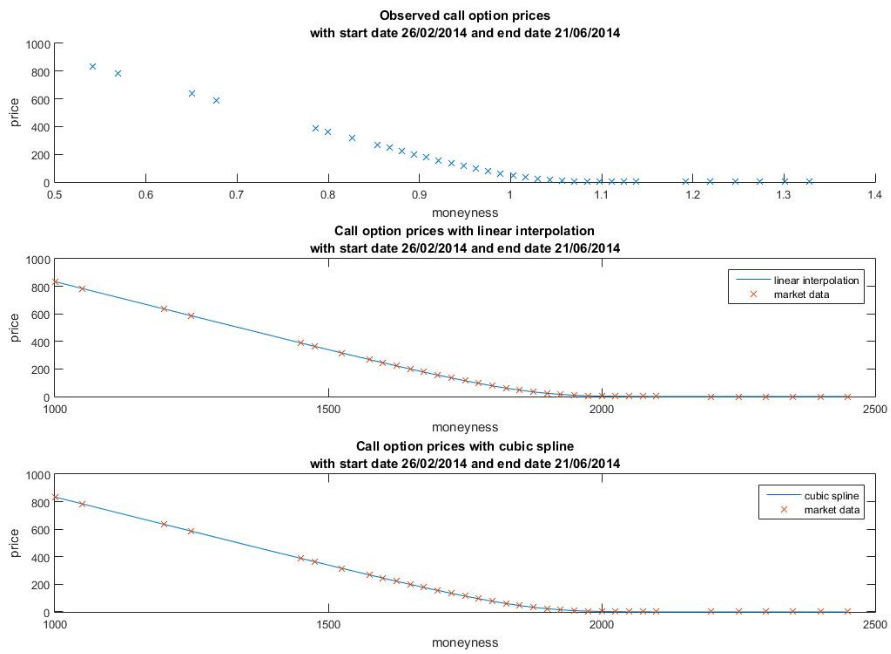

Figure 5: Example of three different call prices on February 26th 2014 expiring on June 21st 2014. The first plots the call prices observed on the market. The second plots the observed data points as well as its linear interpolation. The third plots the observed data points as well as its cubic spline.

Table 2 reports smile-implied deltas for February 26th 2014 computed for three different option contracts expiring on June 21st 2014.

	Raw data (nearest strikes available)	*Linear Interpolation*	*Cubic spline interpolation*
Call #8	0.9503	0.9489	0.9474
Call #18	0.6483	0.6483	0.6479
Call #26	0.0481	0.0481	0.0454

Table 2: Deltas on February 26th 2014 for three different option contracts expiring on June 21st 2014.

4 Results

To analyze the performance of our different delta-hedging strategies, we have calculated the descriptive statistics for the errors, the mean absolute error (MAE) as well as the root mean square error (RMSE) for each contract hedged. The error is defined as the difference between the observed option value and the replicated option value.

For a contract with an initial time to maturity of T days, we have:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{T} error_i^2}{T}} \quad MAE = \frac{\sum_{i=1}^{T} |error_i|}{T}$$

By construction, the RMSE is always greater or equal to the MAE and the larger the difference between these statistics, the larger the variance of the errors associated.

Tables 3 and 4 present the descriptive statistics for the RMSE and MAE for all four methods.

Descriptive statistics RMSE							
Panel A: Call Options							
Model	*Minimum*	*Mean*	*Maximum*	*Standard Deviation*	*Median*	*Skewness*	*Kurtosis*
Practitioners Black-Scholes	0	6.1410	65.6391	6.3820	4.1363	3.0367	17.3912
Smile-Implied Raw data	0	4.3107	22.6761	3.2461	3.7933	1.5959	6.9026
Smile-Implied Linear	0	4.3151	22.6761	3.2387	3.7926	1.5878	6.8835
Smile-Implied Cubic spline	0	4.4166	23.1818	3.2624	3.9016	1.5522	6.7627
Panel B: Put Options							
Model	*Minimum*	*Mean*	*Maximum*	*Standard Deviation*	*Median*	*Skewness*	*Kurtosis*
Practitioners Black-Scholes	0	5.6904	44.4461	7.5201	2.9272	2.2107	8.5091
Smile-Implied Raw data	0	4.7494	38.7440	5.6992	2.9111	2.0013	8.4585
Smile-Implied Linear	0	4.7509	38.7440	5.6938	2.9201	1.9967	8.4393
Smile-Implied Cubic spline	0	4.8029	39.2600	5.7545	2.8898	1.9870	8.3163

Table 3: Descriptive statistics for the RMSE for all four methods during the sample period from September 1st 2013 through August 31st 2014.

Descriptive statistics MAE							
Panel A: Call Options							
Model	*Minimum*	*Mean*	*Maximum*	*Standard Deviation*	*Median*	*Skewness*	*Kurtosis*
Practitioners Black-Scholes	0	5.3408	55.7698	5.8172	3.5058	3.1219	17.6902
Smile-Implied Raw data	0	3.7049	20.9367	2.9473	3.1220	1.7531	7.5936
Smile-Implied Linear	0	3.7093	20.9367	2.9397	3.1386	1.7431	7.5678
Smile-Implied Cubic spline	0	3.7949	21.3793	2.9594	3.1979	1.7051	7.4277
Panel B: Put Options							
Model	*Minimum*	*Mean*	*Maximum*	*Standard Deviation*	*Median*	*Skewness*	*Kurtosis*
Practitioners Black-Scholes	0	5.0143	40.4768	6.7269	2.4724	1.2558	8.7113
Smile-Implied Raw data	0	4.1891	35.4013	5.1056	2.4892	2.1266	9.2702
Smile-Implied Linear	0	4.1902	35.4013	5.1008	2.4818	2.1209	9.2371
Smile-Implied Cubic spline	0	4.2294	35.8880	5.1467	2.4564	2.0995	9.0457

Table 4: Descriptive statistics for the MAE for all four methods during the sample period from September 1st 2013 through August 31st 2014.

We first notice that the model-based approach produces the largest errors. Indeed, the mean is more than 30% higher than under all three other smile-implied methods. The maximum and kurtosis are three times bigger. The replication errors also display a much higher standard deviation and, in a vast majority of cases, a higher skewness as well.

A second observation is that the hedging performance is quite similar across smile-implied methods. Judging by the quality of the S&P500 index options raw data, interpolation appears as a refinement that does not bring much to the approach.

To have a better view on how smile-implied methods perform, we report in Table 5 the average RMSE and MAE in detail across moneyness and maturity.

RMSE								
Panel A: Call Options								
Initial Moneyness	*Practitioners Black-Scholes*		*Smile-Implied Raw data*		*Smile-Implied Linear*		*Smile-Implied Cubic spline*	
	S-T	L-T	S-T	L-T	S-T	L-T	S-T	L-T
ITM	3.5911	6.4487	3.6735	6.5657	3.6714	6.5751	3.7469	6.6469
ATM	4.8672	10.5409	3.4868	5.7678	3.4819	5.7710	3.5832	5.9010
OTM	2.0730	8.2761	0.9969	3.8162	1.0056	3.8477	1.1303	3.9670
Panel B: Put Options								
Initial Moneyness	*Practitioners Black-Scholes*		*Smile-Implied Raw data*		*Smile-Implied Linear*		*Smile-Implied Cubic spline*	
	S-T	L-T	S-T	L-T	S-T	L-T	S-T	L-T
ITM	9.1150	19.2038	5.0366	11.4402	5.0429	11.4919	5.1459	11.7245
ATM	4.4101	7.5176	4.3112	9.3601	4.3120	9.3324	4.3339	9.3524
OTM	0.6367	1.5356	0.2298	0.7896	0.2316	0.7985	0.2798	0.8098
MAE								
Panel A: Call Options								
Initial Moneyness	*Practitioners Black-Scholes*		*Smile-Implied Raw data*		*Smile-Implied Linear*		*Smile-Implied Cubic spline*	
	S-T	L-T	S-T	L-T	S-T	L-T	S-T	L-T
ITM	3.1049	5.5520	3.1777	5.6771	3.1759	5.6861	3.2373	5.7471
ATM	4.1566	9.3859	2.9502	4.9592	2.9457	4.9615	3.0300	5.0729
OTM	1.7872	7.0471	0.8512	3.2996	0.8593	3.3327	0.9664	3.4334
Panel B: Put Options								
Initial Moneyness	*Practitioners Black-Scholes*		*Smile-Implied Raw data*		*Smile-Implied Linear*		*Smile-Implied Cubic spline*	
	S-T	L-T	S-T	L-T	S-T	L-T	S-T	L-T
ITM	8.1691	16.8853	4.3355	9.9245	4.3412	9.9748	4.4292	10.1572
ATM	3.8836	6.5363	3.8311	8.3144	3.8319	8.2870	3.8403	8.3019
OTM	0.5929	1.3739	0.2035	0.7330	0.2052	0.7409	0.2457	0.7462

Table 5: Average RMSE and MAE across the six categories of options for all four methods during the sample period from September 1st 2013 through August 31st 2014. ITM options are options with an initial moneyness level smaller than 0.8 for calls and greater than 1.05 for puts. ATM options have an initial moneyness level between 0.8 and 1.05 included. And OTM options have an initial moneyness level greater than 1.05 for calls and smaller than 0.8 for puts. S-T options are options which live for 90 days or less and L-T options are options that live for more than 90 days.

We observe that the smile-implied method greatly outperforms the benchmark model for OTM options as well as for ATM calls and ITM puts irrespective of their maturity. By contrast the model-free approach and the benchmark model yield a delta hedge of comparable quality when it comes to ITM calls.

5 Conclusion

The smile-implied method is an interesting alternative to standard, model-based delta hedging strategies. Its application on very liquid options such as those written on the S&P500 index yields very promising results.

This work can be extended in several directions. More benchmark models (in particular models accounting for stochastic volatility) could be added into the analysis. In addition, the smile-implied method can readily be extended to managing gamma as well. Finally, the hedging performance could be investigated on other types of underlying assets. For individual equity options for instance, vanilla options are American and their model-based delta is not available in analytical form. This could give an additional edge to the smile-implied method which does not require any estimation.

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