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Price Bounds in Jump-diffusion Markets Revisited via Market Completions

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Price bounds in jump-diffusion markets revisited via market completions

Anne MacKay and Alexander Melnikov

Abstract It is well known that incomplete markets generally admit infinitely many equivalent (local) martingale measures (EMMs), and that the resulting no-arbitrage price of a contingent claim is not unique. The bounds on no-arbitrage prices for a given contingent claim can be obtained by considering the set of EMMs on the market. In some cases, incomplete markets can be completed by adding specific sets of assets. Market completion techniques have been mentioned in various publications and can be used to simplify optimal investment and hedging problems. In this paper, we consider a multidimensional jump-diffusion market with predictable jump sizes and we revisit the no-arbitrage price bounds via market completions. We review the conditions under which a given set of assets can complete the original market, and we present a set of market completions that can be used to obtain the range of no-arbitrage prices.

1 Introduction

The absence of arbitrage in a market allows for the existence of at least one equivalent (local) martingale measure (EMM) that can be used to price contingent claims. In a complete market, such a measure is unique and claims can be perfectly replicated by trading in the available assets; the value of the replicating portfolio is the unique no-arbitrage price of the claim. In an incomplete market, there exists more than one such pricing measure, and some contingent claims cannot be perfectly replicated by investing in the market. These claims do not have a unique no-arbitrage

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price. Instead, there exists a range of no-arbitrage prices that are acceptable for both the buyer and the seller of the claim.

The theory of no-arbitrage pricing in incomplete markets is well developed. An important result is that the range of no-arbitrage prices admitted by an incomplete market can be obtained by taking the infimum and the supremum of the no-arbitrage price over the set of EMMs.

In this paper, we discuss market completions as a way to describe the set of EMMs. The idea of market completions is to associate each equivalent local martingale measure with a set of fictitious assets which forms a complete market when combined with the original incomplete one. The idea of adding fictitious assets to investigate pricing and hedging problems in the Black-Scholes model first appeared in [9], and was used in various works since then. For example, [14] showed how these arguments can be also adapted to American options in a multidimensional diffusion market. Market completion techniques were also independently developed in the framework of multinomial markets in Appendix 3 of [11].

Market completion techniques have also been used to solve precise problems. [6] make external risk tradable via market completion, and use the method to price weather derivatives. In a diffusion model, [10] study pricing and hedging problems by completing a market where incompleteness is due to different borrowing and lending rates (see also [3]). [8] extend the results to a jump-diffusion model. Finally, [4] consider market completion techniques in the setting of a general Lévy model.

In this paper, we consider a multidimensional jump-diffusion model with predictable jump sizes, in which incompleteness stems from a larger number of risk sources than traded assets. We review results on the set of EMMs in this particular model, and describe the assets that can be used to complete the market. We show the equivalence between the set of all EMMs admitted by the market, and a subset of possible market completions. Note that this is only possible because we assume that the jump sizes are predictable (see [12] for details).

The paper is organized as follows. In Section 2, we present the market model and recall specific results on market arbitrage and completeness. We discuss the link between the set of EMMs and market completions in Section 3. Section 4 concludes.

2 Review of basic definitions and concepts

In this section, we introduce the market model and review some key results on no-arbitrage and market completeness conditions in the context of our model.

2.1 Market model

We work on a probability space $(\Omega, \mathscr{F}, P)$ with a finite time horizon $T \in \mathbb{R}$. On the probability space, we assume the existence of a d-dimensional Brownian motion

$W = (W_1, \ldots, W_d)^\top$ and a multivariate Poisson process $N = (N_1, \ldots, N_{n-d})^\top$ with intensity $\lambda = (\lambda_1, \ldots, \lambda_{n-d})^\top$, independent of W. We denote by $\mathbb{F} = \{\mathscr{F}_t\}_{t \le T}$ the filtration generated by W and N, and augmented by the P-null sets. The intensity λ can be stochastic, but is assumed to be predictable in t.

We define a market $(B, S) = (B, S_1, \ldots, S_k)$, in which B denotes the bank account process (considered to be risk-free) and $S = (S_1, \ldots, S_k)$ represents the value of k risky assets. Throughout the paper, we assume that $k \le n$. The $k+1$ assets have the following dynamics

$$\begin{aligned} dB(t) &= B(t)\ r(t)\ dt, \\ dS_i(t) &= S_i(t_-) \left(\mu_i(t)\ dt + \sigma_i^V(t)\ dW(t) + \sigma_i^J(t)\ dM(t)\right) \end{aligned} \tag{1}$$

with $B(0) = 1$ and $S^i(0) = s_0^i \in \mathbb{R}_+$ for $i \in \{1, \ldots, k\}$, and where $M(t) = N(t) - \int_0^t \lambda(s)\ ds$. The risk-free interest rate $r(t) \ge 0$, as well as the appreciation rate $\mu = (\mu_1, \ldots, \mu_k)^\top$ and the matrix-valued processes σ^V and σ^J, with i^{th} row given by the vectors $\sigma_i^V = (\sigma_{i1}^V, \ldots, \sigma_{id}^V)$, and $\sigma_i^J = (\sigma_{i1}^J, \ldots, \sigma_{i(n-d)}^J)$, respectively, for $i = 1, \ldots, k$, are predictable with respect to the filtration $\mathbb{F}$. Going forward, we assume that $r(t) \equiv 0$ for all $t \in [0, T]$, so that $B(t) = 1$ for all $t \in [0, T]$. In other words, we consider that the price processes are discounted by the bank account numéraire.

We also assume that μ, σ^V and σ^J are uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$, and that $\sup_{t \le T} \lambda_l(t) \le K$, for some $K < \infty$, for all $0 \le l \le m$. Under these assumptions, (1) has a unique solution. In order for the risky asset prices to remain positive, we further assume that $\sigma_{il}^J(t) \in [0, 1]$ and $\lambda_l(t) > 0$ and bounded uniformly in t, for $1 \le i \le k$ and $1 \le l \le n - d$.

Finally, we denote by $\sigma = [\sigma^V\ \ \sigma^J]$ the $k \times n$ matrix containing the volatility coefficients. We assume that σ has full rank, so that $\det(\sigma(t)\sigma^\top(t)) \ne 0$ P-a.s. for all $t \in [0, T]$. This allows us to define the process $\theta = [\theta_V\ \ \theta_J]^\top$ by

$$\theta(t) = \sigma^\top(t)(\sigma(t)\sigma^\top(t))^{-1}\mu(t), \tag{2}$$

for $t \in [0, T]$.

We let the $\mathbb{R}^{(k+1)}$-valued process $\pi = (\pi_0(t), \pi_1(t), \ldots, \pi_k(t))_{0 \le t \le T}$ represent a *(portfolio) strategy*, and we assume that $\int_0^T \|\pi(t)\|^2\ dt < \infty$, P-a.s. We denote the value process of the resulting portfolio by X^π, with

$$X^\pi(t) = \pi_0(t)\ B(t) + \sum_{i=1}^k \pi_i(t) S_i(t), \qquad \text{for all } 0 \le t \le T.$$

A portfolio strategy is called *admissible* if its value process satisfies $X^\pi(t) \ge -K$ for some $K = K(\pi) \ge 0$. We denote the class of admissible portfolio strategies with initial capital x by

$$\mathscr{A}(x) = \{\pi \in \mathbb{R}^{k+1} : X^\pi(0) = x,\ X^\pi(t) \ge -K \text{ for all } t \le T\}.$$

An admissible portfolio strategy π is called *self-financing* if the following holds:

$$X^{\pi}(t) = X^{\pi}(0) + \int_0^t \pi_0(t) dB(t) + \sum_{i=1}^{k} \int_0^t \pi_i(t) dS_i(t), \qquad \text{for all } 0 \leq t \leq T. \quad (3)$$

Note that under our assumption that $r(t) \equiv 0$, the second term on the right-hand side of (3) is always 0, and will therefore be omitted going forward.

As is usually the case, we only consider arbitrage-free markets. That is, we assume that for any self-financing strategy π in $\mathscr{A}(0)$, for all $0 \leq t \leq T$,

$$P(X^{\pi}(t) = 0) = 1, \qquad P(X^{\pi}(t) \geq 0) = 1 \qquad \Rightarrow \qquad P(X^{\pi}(t) = 0) = 1.$$

The existence of an equivalent martingale measure (EMM), i.e. a measure equivalent to P under which the value of any self-financing strategy is a local martingale, is a sufficient condition for our market to be arbitrage-free. Such a measure exists if the market allows for at least one predictable process $\gamma = (\gamma^V, \gamma^J)^\top$ with $\gamma^J = (\gamma_1^J, \ldots, \gamma_{n-d}^J)$ strictly positive that satisfies

$$\sigma^V(t)\gamma^V(t) + \sigma^J(t)\lambda(t) \bullet (\mathbf{1} - \gamma^J(t)) = \mu(t) = \sigma(t)\theta(t), \quad (4)$$

where $\mathbf{1}$ denotes a vector of ones, and where the second equality follows from (2). Heuristically, this condition ensures that the drift of the discounted asset prices "disappears" under an EMM. It is analogous to the no-arbitrage condition in a multidimensional diffusion market, and comes from similar arguments. Going forward, we will assume the existence of at least one process γ as described above.

It is possible to show that any solution γ to (4), with $\gamma_l^J > 0$ for $l \in \{1, \ldots, n-d\}$, defines a probability measure. Indeed, for such a solution γ, we can define the process $L_\gamma = L_\gamma^V L_\gamma^J$, with

$$L_\gamma^V(t) = \exp\left\{ -\int_0^t \gamma^V(s)^\top dW(s) - \frac{1}{2}\int_0^t \|\gamma^V(s)\|^2 ds \right\},$$
$$L_\gamma^J(t) = \exp\left\{ -\int_0^t \lambda(s) \bullet (1 - \gamma^J(s)) ds \right\} \prod_{l=1}^{n-d} \prod_{s \leq t} \gamma_l^J(s) \Delta N_l(s), \qquad \text{for } t \leq T.$$

Then L_γ is a non-negative local martingale with $E[L_\gamma(t)] = 1$ for all $t \in [0, T]$. We are only interested in the solutions γ such that L_γ is a true martingale, and we define this set by

$$\Gamma = \{\gamma : \gamma \text{ solves (4)}, \gamma_l^J > 0 \text{ for } l \in \{1, \ldots, n-d\}, L_\gamma \text{ is a martingale}\}.$$

It is well known by now that the set Γ characterizes the set of all possible EMMs on the (B, S) market (see for example [2]). This result is summarized in the following proposition.

Proposition 2.1 (Theorem 4.2 of [2]) *Let $\mathscr{Q}$ denote the set of all EMMs on the (B,S) market and let $\left.\frac{dQ_\gamma}{dP}\right|_{\mathscr{F}_t} = L_\gamma(t)$. Then, $Q_\gamma \in \mathscr{Q}$ if and only if $\gamma \in \Gamma$.*

The reader is referred to [2] for the proof of this proposition.

2.2 Market (in)completeness and hedging contingent claims

Jumps in the stock price process are often a source of incompleteness. In our particular case, the size of the jumps is predictable, and the market is incomplete only when there are less assets than sources of risk. Next, we review the concept of market (in)completeness and discuss it in the context of the (B,S) market.

Market completeness is closely linked to the perfect replication of contingent claims. We define a *contingent claim* as an $\mathscr{F}_T$-measurable random variable $f_T = f_T(\omega)$ that satisfies $\mathbb{E}_Q[f_T] < \infty$ for all $Q \in \mathscr{Q}$.

A contingent claim is called *replicable* if there exists an initial capital x and an admissible, self-financing strategy π that satisfies

$$X^{\pi}(T) = x + \sum_{i=1}^{k} \int_0^T \pi_i(t) dS_i(t) = f_T, \qquad P - \text{a.s.}$$

A market is called *complete* if any contingent claim is replicable. In a complete market, the *perfect hedging price* (or *fair price*) $\mathbf{C}(f_T)$ of a replicable contingent claim f_T is defined as the lowest initial capital needed to perfectly replicate the claim at T:

$$\mathbf{C}(f_T, P) = \inf\{x \geq 0 : \exists\, \pi \in \mathscr{A}(x) \text{ s.t. } X_T^{\pi} = f_T, P - \text{a.s.}\}.$$

The perfect hedging price is also obtained as the expectation of the (discounted) claim under the unique EMM, that is

$$\mathbf{C}(f_T) = \mathbb{E}_Q[f_T]. \tag{5}$$

A well-known necessary and sufficient condition for market completeness is the uniqueness of the EMM. In our setting, this is equivalent to the set Γ being a singleton. In other words, if (4) has only one solution such that $\gamma_l^J > 0$ for $l \in \{1, \ldots, n-d\}$ and L_γ is a martingale, then the market is complete.

Note that (4) can be written as a system of k equations, and γ is a vector of length n. Thus, when $k < n$, the solution cannot be unique. It is only possible for our market to be complete when $k = n$, that is, when there are as many assets as sources of risk.

In the case where $k = n$, the unique element of Γ is given by $\gamma^V(t) = \theta^V(t)$ and $\gamma^J(t) = \lambda^{-1}(t) \bullet (\lambda(t) - \theta^J(t))$, if $\theta^J(t) < \lambda(t)$ for all $t \in [0,T]$. When the bound on $\theta^J(t)$ is not satisfied, the market does not admit any martingale measure. This result is discussed, for example, in [1] and [7].

In this paper, our goal is to study no-arbitrage price bounds in incomplete markets. Henceforth, we assume $k < n$, which results in market incompleteness.

As recalled previously, in an incomplete market, some contingent claims are not perfectly replicable. That is, it is impossible to find a self-financing admissible trading strategy whose value at T is equal to f_T P-almost surely. Therefore, we extend the set of admissible strategies to consider investment strategies with consumption. Such strategies will be represented by a $(k+2)$-dimensional $\mathbb{F}$-adapted process $(\pi, c) = (\pi_0(t), \pi_1(t), \ldots, \pi_k(t), c(t))_{t\leq T}$, where $c(t) \geq 0$ for $t \leq T$. The value process of the strategy (π, c) is given by

$$X^{\pi,c}(t) = X^{\pi,c}(0) + \sum_{i=1}^{k} \int_0^t \pi_i(s)dS_i(s) - \int_0^t c(s)ds.$$

The strategy (π, c) with initial capital x is a (super-)hedge for the contingent claim f_T if its value process satisfies $X^{\pi,c}(T) \geq f_T$, $P-$a.s..

An investor selling the contingent claim f_T will require its price to be at least sufficient to build a (super-)hedging portfolio for the claim. Thus, in the (B, S) market, we call the *upper hedging price* (or *seller price*) $\mathbf{C}^*(f_T)$ the smallest initial capital needed by the investor to set up such a portfolio for f_T:

$$\mathbf{C}^*(f_T) = \inf\{x \geq 0 : \exists(\pi, c) \in \mathscr{A}(x) : X^{\pi,c}(T) \geq f_T, P-\text{a.s.}\} \tag{6}$$

An investor buying the contingent claim f_T will not want to pay more than the amount that she will be able to recover by time T, by investing in a strategy with consumption. Therefore, the largest amount allowing for such result is called *lower hedging price* (or *buyer price*) $\mathbf{C}_*(f_T)$ is given by:

$$\mathbf{C}_*(f_T) = \inf\{x \geq 0 : \exists(\pi, c) \in \mathscr{A}(-x) : X^{\pi,c}(T) \geq -f_T, P-\text{a.s.}\} \tag{7}$$

Claims that cannot be perfectly replicated in an incomplete market do not have a unique, perfect hedging price as defined by (5). Indeed, each measure $Q \in \mathscr{Q}$ yields a different arbitrage price $\mathbb{E}_Q[f_T]$. It is well known that the lower and upper hedging prices correspond to the lower and upper bounds of the set of arbitrage prices (for more details on this result, the reader is referred to [5]).

The upper and lower hedging prices for a contingent claim f_T can thus be obtained by taking the infimum and the supremum over the set of EMMs admitted by an incomplete market:

$$\mathbf{C}^*(f_T) = \sup_{Q\in\mathscr{Q}} \mathbb{E}_Q[f_T], \qquad \mathbf{C}_*(f_T) = \inf_{Q\in\mathscr{Q}} \mathbb{E}_Q[f_T].$$

3 Pricing via market completions

In this section, we show how the upper and lower hedging prices can be represented in terms of market completions, or additional assets added to complete the original (B, S) market. In particular, in the market defined in Section 2.1, we characterize the set of completions that should be considered to obtain the no-arbitrage price bounds.

3.1 Market completions

Let S^c be an $(n-k)$-dimensional process representing the value of $(n-k)$ assets that will be added to the original market. We assume that the new assets have dynamics similar to the first k ones, that is, for $i \in \{k+1,\dots,n\}$,

$$dS_i(t) = S_i(t_-)\left(\nu_i(t)\ dt + \rho_i^V(t)\ dW(t) + \rho_i^J(t)\ dM(t)\right),$$

with $S_i(0) = s_0^i \in \mathbb{R}_+$, and where W and M are defined as in Section 2.1. The $(n-k)$-dimensional appreciation rate process ν and the matrix-valued processes ρ^V and ρ^J, with i^{th} row given by $\rho_i^V = (\rho_{i1}^V,\dots,\rho_{id}^V)$, and $\rho_i^J = (\rho_{i1}^J,\dots,\rho_{i(n-d)}^J)$, respectively, for $i = 1,\dots,n-k$, are predictable with respect to the filtration $\mathbb{F}$. As in the original market, we assume that ν and $\rho = [\rho^V\ \rho^J]$ are uniformly bounded in $(t,\omega) \in [0,T]\times\Omega$, and that $\rho_{il}^J \in [0,1]$, for $k+1 \le i \le n$ and $1 \le l \le n-d$.

We denote $\tilde{S} = (S,S^c)$, $\tilde{\mu} = (\mu,\nu)^\top$ and $\tilde{\sigma} = \binom{\sigma}{\rho}$, and only consider completions S^c such that the augmented market $(B,\tilde{S})$ is complete. That is, for a given market augmented with the completion S^c, we assume that $\det\tilde{\sigma} \neq 0$, and we define $\widetilde{\theta} = (\widetilde{\theta}^V,\widetilde{\theta}^J)$ by

$$\widetilde{\theta} = \widetilde{\sigma}^\top\left(\widetilde{\sigma}\widetilde{\sigma}^\top\right)^{-1}\widetilde{\mu}, \tag{8}$$

and $\widetilde{\gamma} = (\widetilde{\gamma}^V,\widetilde{\gamma}^J)$ by

$$\tilde{\gamma}^V = \tilde{\theta}^V \qquad\qquad \tilde{\gamma}^J = \lambda^{-1}\bullet\left(\lambda - \tilde{\theta}^J\right). \tag{9}$$

If $\widetilde{\theta}^J(t) < \lambda(t)$ for all $t \in [0,T]$, then $L_{\widetilde{\gamma}}$ is a true martingale and the $(B,\widetilde{S})$ market is complete.

Then, $Q_{\widetilde{\gamma}}$, defined by $\frac{dQ_{\tilde{\gamma}}}{dP} = L_{\tilde{\gamma}}$, is the unique EMM on the completed market $(B,\widetilde{S})$. It immediately follows that a contingent claim f_T has the unique no-arbitrage price $E[L_{\widetilde{\gamma}}f_T]$ in the completed market.

3.2 A special set of market completions

In this section, we want to express the upper and lower hedging prices defined in (6) and (7) in terms of market completions. To do so, we follow ideas similar to those used by [14] in a multivariate diffusion setting.

We let $\mathscr{R}_\rho$ be defined as the set of $(n-k)\times n$ matrix-valued $\mathbb{F}$-adapted process uniformly bounded in $(t,\omega) \in [0,T]\times\Omega$, with $\rho_{il}^J \in [0,1]$, for $k+1 \le i \le n$ and $1 \le l \le n-d$, such that $\det\binom{\sigma}{\rho} \neq 0$ P-a.s.

For each $\rho \in \mathscr{R}_\rho$, we denote by $\mathscr{D}_\rho$ the set of appreciation rate processes ν for which the associated market is complete. Therefore, we have

$$\mathscr{D}_\rho := \{\nu : \nu \text{ is } \mathbb{F}\text{-predictable, uniformly bounded, s.t. } \widetilde{\theta}^J(t) < \lambda(t)\ \forall\, t \in [0,T]\}.$$

For a given $\rho \in \mathscr{R}_\rho$, we define the *upper* and *lower completion prices* $\widetilde{\mathbf{C}}^*(f_T;\rho)$ and $\widetilde{\mathbf{C}}_*(f_T;\rho)$ by

$$\widetilde{\mathbf{C}}^*(f_T;\rho) = \sup_{\nu\in\mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(\nu,\rho)f_T], \qquad \widetilde{\mathbf{C}}_*(f_T;\rho) = \inf_{\nu\in\mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(\nu,\rho)f_T].$$

In the above, the density process $L_{\widetilde{\gamma}}(\nu,\rho) = L^V_{\widetilde{\gamma}}(\nu,\rho)L^J_{\widetilde{\gamma}}(\nu,\rho)$ defines the unique EMM on the market completed using assets with appreciation rate vector ν and diffusion coefficient matrix ρ.

As it is the case in the multidimensional diffusion market (see [14]), the upper and lower completion prices do not depend on the choice of ρ.

Proposition 3.1 *Fix ρ and $\rho' \in \mathscr{R}_\rho$. Then,*

$$\widetilde{\mathbf{C}}^*(f_T;\rho) = \widetilde{\mathbf{C}}^*(f_T;\rho'), \qquad \widetilde{\mathbf{C}}_*(f_T;\rho) = \widetilde{\mathbf{C}}_*(f_T;\rho').$$

Proof. The proof is very similar to the proof of Proposition 2.1 of [14]. Take the $(n-k)\times k$ and $(n-k)\times(n-k)$ predictable matrix valued processes C and D with $\det(D) \neq 0$ satisfying

$$\begin{pmatrix}\sigma\\ \rho'\end{pmatrix} = \begin{pmatrix}I & 0\\ C & D\end{pmatrix}\begin{pmatrix}\sigma\\ \rho\end{pmatrix},$$

where I denotes the identity matrix. Then it is possible to show that

$$\widetilde{\theta}_{\nu,\rho'} = \begin{pmatrix}\sigma\\ \rho'\end{pmatrix}^\top \left(\begin{pmatrix}\sigma\\ \rho'\end{pmatrix}\begin{pmatrix}\sigma\\ \rho'\end{pmatrix}^\top\right)^{-1}\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \widetilde{\theta}_{\nu',\rho},$$

with $\nu' = D^{-1}(\nu - C\mu) \in \mathscr{D}_\rho$. The result follows. □

Since the upper and lower completion prices are independent of ρ, it is natural to only consider the market completions associated with a particular matrix-valued process ρ. Henceforth, we fix $\bar{\rho} \in \mathscr{R}_\rho$ satisfying

$$\sigma\bar{\rho}^\top = 0 \qquad \text{and} \qquad \bar{\rho}\bar{\rho}^\top = I. \tag{10}$$

It follows that $\widetilde{\theta}_{\nu,\bar{\rho}} = \theta + \vartheta_\nu$ with $\vartheta_\nu = \bar{\rho}^\top(\bar{\rho}\bar{\rho}^\top)^{-1}\nu$ for any $\nu \in \mathscr{D}_{\bar{\rho}}$.

The density process $L_{\widetilde{\gamma}}(\nu,\bar{\rho})$ of the EMM associated with each market completion with parameters $\bar{\rho}$ and $\nu \in \mathscr{D}_{\bar{\rho}}$ can then be written as

$$\begin{aligned}L_{\widetilde{\gamma}}(t;\nu,\bar{\rho}) =& e^{-\int_0^t \gamma^V(s)^\top dW(s) - \int_0^t \vartheta^V_\nu(s)^\top dW(s) - \frac{1}{2}\int_0^t \|\theta^V(s)+\vartheta^V_\nu(s)\|^2 ds}\\ &\times e^{-\int_0^t \theta^J(s)\, ds - \int_0^t \vartheta^J_\nu(s)\, ds}\prod_{l=1}^{n-d}\prod_{s\le t}\lambda^{-1}(s)\bullet\left(\lambda(s)-\theta^J(s)-\vartheta^J_\nu(s)\right)\Delta N_l(s).\end{aligned}$$

In the above, $\theta = (\theta^V,\theta^J)$ is as defined in (2), and is therefore independent of the market completion.

3.3 Completion price bounds

Finally, we highlight the equivalence between the set of EMMs $\mathscr{Q}$ of the original market and the set of market completions associated with the fixed matrix process ρ satisfying (10).

Lemma 1. *Fix $\rho \in \mathscr{R}_\rho$. Then for any $\gamma \in \Gamma$, it is possible to find $\nu \in \mathscr{D}_\rho$ such that*

$$\gamma^V = \tilde{\theta}^V_{\nu,\rho}, \qquad \gamma^J = \lambda^{-1} \bullet (\lambda - \widetilde{\theta}^J_{\nu,\rho}).$$

Proof. To find such a $\nu \in \mathscr{D}_\rho$, it suffices to let $\widetilde{\theta}^V = \gamma^V$ and $\widetilde{\theta}^J = \lambda - \lambda \bullet \gamma^J$. Then, since $\widetilde{\sigma}^\top(\widetilde{\sigma}\widetilde{\sigma}^\top)^{-1}$ is an $n \times n$ matrix, there is only one solution $\widetilde{\mu} = (\mu, \nu)$ that satisfies

$$\widetilde{\theta} = \widetilde{\sigma}^\top(\widetilde{\sigma}\widetilde{\sigma}^\top)^{-1}\widetilde{\mu}. \tag{11}$$

Indeed, this solution is given by $\widetilde{\mu} = \widetilde{\sigma}\widetilde{\theta}$, and $\nu \in \mathscr{D}_\rho$ by definition. □

It follows from Lemma 1 that any measure $Q \in \mathscr{Q}$ can be recovered by completing the market using a market completion S^c with parameters $\bar{\rho}$ and ν, for some $\nu \in \mathscr{D}_{\bar{\rho}}$.

We can also show that any $\nu \in \mathscr{D}_{\bar{\rho}}$ defines an element $\gamma \in \Gamma$.

Lemma 2. *Fix $\rho \in \mathscr{R}_\rho$, with ρ satisfying* (10). *Then for any $\nu \in \mathscr{D}_\rho$, the resulting $\widetilde{\gamma}$, as defined by* (8) *and* (9) *is an element of Γ.*

Proof. From (10), we have $\widetilde{\theta} = \theta + \vartheta$, and the resulting $\tilde{\gamma}$ solves (4), since $\sigma\theta^\top = \mu$ and $\sigma\vartheta^\top = 0$. □

Therefore, the unique EMM resulting from any market completion S^c with parameters $\bar{\rho}$ and ν, with $\nu \in \mathscr{D}_{\bar{\rho}}$ is an element of $\mathscr{Q}$.

Lemmas 1 and 2 confirm that the set of market completions with diffusion coefficient matrix ρ fixed spans the set of EMMs $\mathscr{Q}$ on the complete market. It is therefore possible to express the range of no-arbitrage prices for a contingent claim f_T in terms of the set of market completions.

Proposition 3.2 *The upper and lower hedging prices $\mathbf{C}^*(f_T)$ and $\mathbf{C}_*(f_T)$ coincide with the upper and lower completion prices $\tilde{\mathbf{C}}^*(f_T)$ and $\tilde{\mathbf{C}}_*(f_T)$, and we have*

$$\mathbf{C}^*(f_T) = \sup_{\nu \in \mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(T;\nu,\rho) f_T], \qquad \mathbf{C}_*(f_T) = \inf_{\nu \in \mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(T;\nu,\rho) f_T].$$

4 Concluding remarks

In a pure diffusion market, the process $L_{\widetilde{\gamma}}(\nu,\rho)$ can be expressed as the product of two (local) martingales; one pertaining to the original market, and the other one associated with the market completion. This makes market completion techniques very useful in the context of hedging and portfolio optimization problems. The addition of jumps in the market generally removes the possibility of such an expression, as is remarked at the end of Section 1 of [13].

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Price bounds in jump-diffusion markets revisited via market completions

Anne MacKay and Alexander Melnikov

Abstract It is well known that incomplete markets generally admit infinitely many equivalent (local) martingale measures (EMMs), and that the resulting no-arbitrage price of a contingent claim is not unique. The bounds on no-arbitrage prices for a given contingent claim can be obtained by considering the set of EMMs on the market. In some cases, incomplete markets can be completed by adding specific sets of assets. Market completion techniques have been mentioned in various publications and can be used to simplify optimal investment and hedging problems. In this paper, we consider a multidimensional jump-diffusion market with predictable jump sizes and we revisit the no-arbitrage price bounds via market completions. We review the conditions under which a given set of assets can complete the original market, and we present a set of market completions that can be used to obtain the range of no-arbitrage prices.

1 Introduction

The absence of arbitrage in a market allows for the existence of at least one equivalent (local) martingale measure (EMM) that can be used to price contingent claims. In a complete market, such a measure is unique and claims can be perfectly replicated by trading in the available assets; the value of the replicating portfolio is the unique no-arbitrage price of the claim. In an incomplete market, there exists more than one such pricing measure, and some contingent claims cannot be perfectly replicated by investing in the market. These claims do not have a unique no-arbitrage

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price. Instead, there exists a range of no-arbitrage prices that are acceptable for both the buyer and the seller of the claim.

The theory of no-arbitrage pricing in incomplete markets is well developed. An important result is that the range of no-arbitrage prices admitted by an incomplete market can be obtained by taking the infimum and the supremum of the no-arbitrage price over the set of EMMs.

In this paper, we discuss market completions as a way to describe the set of EMMs. The idea of market completions is to associate each equivalent local martingale measure with a set of fictitious assets which forms a complete market when combined with the original incomplete one. The idea of adding fictitious assets to investigate pricing and hedging problems in the Black-Scholes model first appeared in [9], and was used in various works since then. For example, [14] showed how these arguments can be also adapted to American options in a multidimensional diffusion market. Market completion techniques were also independently developed in the framework of multinomial markets in Appendix 3 of [11].

Market completion techniques have also been used to solve precise problems. [6] make external risk tradable via market completion, and use the method to price weather derivatives. In a diffusion model, [10] study pricing and hedging problems by completing a market where incompleteness is due to different borrowing and lending rates (see also [3]). [8] extend the results to a jump-diffusion model. Finally, [4] consider market completion techniques in the setting of a general Lévy model.

In this paper, we consider a multidimensional jump-diffusion model with predictable jump sizes, in which incompleteness stems from a larger number of risk sources than traded assets. We review results on the set of EMMs in this particular model, and describe the assets that can be used to complete the market. We show the equivalence between the set of all EMMs admitted by the market, and a subset of possible market completions. Note that this is only possible because we assume that the jump sizes are predictable (see [12] for details).

The paper is organized as follows. In Section 2, we present the market model and recall specific results on market arbitrage and completeness. We discuss the link between the set of EMMs and market completions in Section 3. Section 4 concludes.

2 Review of basic definitions and concepts

In this section, we introduce the market model and review some key results on no-arbitrage and market completeness conditions in the context of our model.

2.1 Market model

We work on a probability space $(\Omega, \mathscr{F}, P)$ with a finite time horizon $T \in \mathbb{R}$. On the probability space, we assume the existence of a d-dimensional Brownian motion

$W = (W_1, \ldots, W_d)^\top$ and a multivariate Poisson process $N = (N_1, \ldots, N_{n-d})^\top$ with intensity $\lambda = (\lambda_1, \ldots, \lambda_{n-d})^\top$, independent of W. We denote by $\mathbb{F} = \{\mathscr{F}_t\}_{t \leq T}$ the filtration generated by W and N, and augmented by the P-null sets. The intensity λ can be stochastic, but is assumed to be predictable in t.

We define a market $(B, S) = (B, S_1, \ldots, S_k)$, in which B denotes the bank account process (considered to be risk-free) and $S = (S_1, \ldots, S_k)$ represents the value of k risky assets. Throughout the paper, we assume that $k \leq n$. The $k+1$ assets have the following dynamics

$$
\begin{aligned}
dB(t) &= B(t)\, r(t)\, dt, \\
dS_i(t) &= S_i(t_-) \left(\mu_i(t)\, dt + \sigma_i^V(t)\, dW(t) + \sigma_i^J(t)\, dM(t) \right)
\end{aligned} \tag{1}
$$

with $B(0) = 1$ and $S^i(0) = s_0^i \in \mathbb{R}_+$ for $i \in \{1, \ldots, k\}$, and where $M(t) = N(t) - \int_0^t \lambda(s)\, ds$. The risk-free interest rate $r(t) \geq 0$, as well as the appreciation rate $\mu = (\mu_1, \ldots, \mu_k)^\top$ and the matrix-valued processes σ^V and σ^J, with i^{th} row given by the vectors $\sigma_i^V = (\sigma_{i1}^V, \ldots, \sigma_{id}^V)$, and $\sigma_i^J = (\sigma_{i1}^J, \ldots, \sigma_{i(n-d)}^J)$, respectively, for $i = 1, \ldots, k$, are predictable with respect to the filtration $\mathbb{F}$. Going forward, we assume that $r(t) \equiv 0$ for all $t \in [0, T]$, so that $B(t) = 1$ for all $t \in [0, T]$. In other words, we consider that the price processes are discounted by the bank account numéraire.

We also assume that μ, σ^V and σ^J are uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$, and that $\sup_{t \leq T} \lambda_l(t) \leq K$, for some $K < \infty$, for all $0 \leq l \leq m$. Under these assumptions, (1) has a unique solution. In order for the risky asset prices to remain positive, we further assume that $\sigma_{il}^J(t) \in [0, 1]$ and $\lambda_l(t) > 0$ and bounded uniformly in t, for $1 \leq i \leq k$ and $1 \leq l \leq n - d$.

Finally, we denote by $\sigma = [\sigma^V \ \sigma^J]$ the $k \times n$ matrix containing the volatility coefficients. We assume that σ has full rank, so that $\det(\sigma(t)\sigma^\top(t)) \neq 0$ P-a.s. for all $t \in [0, T]$. This allows us to define the process $\theta = [\theta_V \ \theta_J]^\top$ by

$$
\theta(t) = \sigma^\top(t)(\sigma(t)\sigma^\top(t))^{-1}\mu(t), \tag{2}
$$

for $t \in [0, T]$.

We let the $\mathbb{R}^{(k+1)}$-valued process $\pi = (\pi_0(t), \pi_1(t), \ldots, \pi_k(t))_{0 \leq t \leq T}$ represent a *(portfolio) strategy*, and we assume that $\int_0^T \|\pi(t)\|^2\, dt < \infty$, P-a.s. We denote the value process of the resulting portfolio by X^π, with

$$
X^\pi(t) = \pi_0(t)\, B(t) + \sum_{i=1}^{k} \pi_i(t) S_i(t), \qquad \text{for all } 0 \leq t \leq T.
$$

A portfolio strategy is called *admissible* if its value process satisfies $X^\pi(t) \geq -K$ for some $K = K(\pi) \geq 0$. We denote the class of admissible portfolio strategies with initial capital x by

$$
\mathscr{A}(x) = \{\pi \in \mathbb{R}^{k+1} : X^\pi(0) = x,\ X^\pi(t) \geq -K \text{ for all } t \leq T\}.
$$

An admissible portfolio strategy π is called *self-financing* if the following holds:

$$X^{\pi}(t)=X^{\pi}(0)+\int_0^t \pi_0(t)dB(t)+\sum_{i=1}^{k}\int_0^t \pi_i(t)dS_i(t), \qquad \text{for all } 0\le t\le T. \quad (3)$$

Note that under our assumption that $r(t)\equiv 0$, the second term on the right-hand side of (3) is always 0, and will therefore be omitted going forward.

As is usually the case, we only consider arbitrage-free markets. That is, we assume that for any self-financing strategy π in $\mathscr{A}(0)$, for all $0\le t\le T$,

$$P(X^{\pi}(t)=0)=1, \qquad P(X^{\pi}(t)\ge 0)=1 \qquad \Rightarrow \qquad P(X^{\pi}(t)=0)=1.$$

The existence of an equivalent martingale measure (EMM), i.e. a measure equivalent to P under which the value of any self-financing strategy is a local martingale, is a sufficient condition for our market to be arbitrage-free. Such a measure exists if the market allows for at least one predictable process $\gamma=(\gamma^V,\gamma^J)^\top$ with $\gamma^J=(\gamma^J_1,\dots,\gamma^J_{n-d})$ strictly positive that satisfies

$$\sigma^V(t)\gamma^V(t)+\sigma^J(t)\lambda(t)\bullet(\mathbf{1}-\gamma^J(t))=\mu(t)=\sigma(t)\theta(t), \quad (4)$$

where $\mathbf{1}$ denotes a vector of ones, and where the second equality follows from (2). Heuristically, this condition ensures that the drift of the discounted asset prices "disappears" under an EMM. It is analogous to the no-arbitrage condition in a multidimensional diffusion market, and comes from similar arguments. Going forward, we will assume the existence of at least one process γ as described above.

It is possible to show that any solution γ to (4), with $\gamma^J_l>0$ for $l\in\{1,\dots,n-d\}$, defines a probability measure. Indeed, for such a solution γ, we can define the process $L_\gamma=L^V_\gamma L^J_\gamma$, with

$$L^V_\gamma(t)=\exp\left\{-\int_0^t\gamma^V(s)^\top dW(s)-\frac{1}{2}\int_0^t\|\gamma^V(s)\|^2ds\right\},$$
$$L^J_\gamma(t)=\exp\left\{-\int_0^t\lambda(s)\bullet(1-\gamma^J(s))ds\right\}\prod_{l=1}^{n-d}\prod_{s\le t}\gamma^J_l(s)\Delta N_l(s), \qquad \text{for } t\le T.$$

Then L_γ is a non-negative local martingale with $E[L_\gamma(t)]=1$ for all $t\in[0,T]$. We are only interested in the solutions γ such that L_γ is a true martingale, and we define this set by

$$\Gamma=\{\gamma:\gamma \text{ solves (4)}, \gamma^J_l>0 \text{ for } l\in\{1,\dots,n-d\}, L_\gamma \text{ is a martingale}\}.$$

It is well known by now that the set Γ characterizes the set of all possible EMMs on the (B,S) market (see for example [2]). This result is summarized in the following proposition.

Proposition 2.1 (Theorem 4.2 of [2]) *Let $\mathscr{Q}$ denote the set of all EMMs on the (B,S) market and let $\frac{dQ_\gamma}{dP}\Big|_{\mathscr{F}_t}=L_\gamma(t)$. Then, $Q_\gamma\in\mathscr{Q}$ if and only if $\gamma\in\Gamma$.*

The reader is referred to [2] for the proof of this proposition.

2.2 Market (in)completeness and hedging contingent claims

Jumps in the stock price process are often a source of incompleteness. In our particular case, the size of the jumps is predictable, and the market is incomplete only when there are less assets than sources of risk. Next, we review the concept of market (in)completeness and discuss it in the context of the (B,S) market.

Market completeness is closely linked to the perfect replication of contingent claims. We define a *contingent claim* as an $\mathscr{F}_T$-measurable random variable $f_T = f_T(\omega)$ that satisfies $\mathbb{E}_Q[f_T] < \infty$ for all $Q \in \mathscr{Q}$.

A contingent claim is called *replicable* if there exists an initial capital x and an admissible, self-financing strategy π that satisfies

$$X^\pi(T) = x + \sum_{i=1}^{k} \int_0^T \pi_i(t) dS_i(t) = f_T, \qquad P - \text{a.s.}$$

A market is called *complete* if any contingent claim is replicable. In a complete market, the *perfect hedging price* (or *fair price*) $\mathbf{C}(f_T)$ of a replicable contingent claim f_T is defined as the lowest initial capital needed to perfectly replicate the claim at T:

$$\mathbf{C}(f_T, P) = \inf\{x \geq 0 : \exists\, \pi \in \mathscr{A}(x) \text{ s.t. } X_T^\pi = f_T, P - \text{a.s.}\}.$$

The perfect hedging price is also obtained as the expectation of the (discounted) claim under the unique EMM, that is

$$\mathbf{C}(f_T) = \mathbb{E}_Q[f_T]. \tag{5}$$

A well-known necessary and sufficient condition for market completeness is the uniqueness of the EMM. In our setting, this is equivalent to the set Γ being a singleton. In other words, if (4) has only one solution such that $\gamma_l^J > 0$ for $l \in \{1, \ldots, n-d\}$ and L_γ is a martingale, then the market is complete.

Note that (4) can be written as a system of k equations, and γ is a vector of length n. Thus, when $k < n$, the solution cannot be unique. It is only possible for our market to be complete when $k = n$, that is, when there are as many assets as sources of risk.

In the case where $k = n$, the unique element of Γ is given by $\gamma^V(t) = \theta^V(t)$ and $\gamma^J(t) = \lambda^{-1}(t) \bullet (\lambda(t) - \theta^J(t))$, if $\theta^J(t) < \lambda(t)$ for all $t \in [0,T]$. When the bound on $\theta^J(t)$ is not satisfied, the market does not admit any martingale measure. This result is discussed, for example, in [1] and [7].

In this paper, our goal is to study no-arbitrage price bounds in incomplete markets. Henceforth, we assume $k < n$, which results in market incompleteness.

As recalled previously, in an incomplete market, some contingent claims are not perfectly replicable. That is, it is impossible to find a self-financing admissible trading strategy whose value at T is equal to f_T P-almost surely. Therefore, we extend the set of admissible strategies to consider investment strategies with consumption. Such strategies will be represented by a $(k+2)$-dimensional $\mathbb{F}$-adapted process $(\pi,c)=(\pi_0(t),\pi_1(t),\ldots,\pi_k(t),c(t))_{t\le T}$, where $c(t)\ge 0$ for $t\le T$. The value process of the strategy (π,c) is given by

$$X^{\pi,c}(t)=X^{\pi,c}(0)+\sum_{i=1}^{k}\int_0^t \pi_i(s)dS_i(s)-\int_0^t c(s)ds.$$

The strategy (π,c) with initial capital x is a (super-)hedge for the contingent claim f_T if its value process satisfies $X^{\pi,c}(T)\ge f_T$, $P-$a.s..

An investor selling the contingent claim f_T will require its price to be at least sufficient to build a (super-)hedging portfolio for the claim. Thus, in the (B,S) market, we call the *upper hedging price* (or *seller price*) $\mathbf{C}^*(f_T)$ the smallest initial capital needed by the investor to set up such a portfolio for f_T:

$$\mathbf{C}^*(f_T)=\inf\{x\ge 0:\exists(\pi,c)\in\mathscr{A}(x):X^{\pi,c}(T)\ge f_T,P-\text{a.s.}\} \tag{6}$$

An investor buying the contingent claim f_T will not want to pay more than the amount that she will be able to recover by time T, by investing in a strategy with consumption. Therefore, the largest amount allowing for such result is called *lower hedging price* (or *buyer price*) $\mathbf{C}_*(f_T)$ is given by:

$$\mathbf{C}_*(f_T)=\inf\{x\ge 0:\exists(\pi,c)\in\mathscr{A}(-x):X^{\pi,c}(T)\ge -f_T,P-\text{a.s.}\} \tag{7}$$

Claims that cannot be perfectly replicated in an incomplete market do not have a unique, perfect hedging price as defined by (5). Indeed, each measure $Q\in\mathscr{Q}$ yields a different arbitrage price $\mathbb{E}_Q[f_T]$. It is well known that the lower and upper hedging prices correspond to the lower and upper bounds of the set of arbitrage prices (for more details on this result, the reader is referred to [5]).

The upper and lower hedging prices for a contingent claim f_T can thus be obtained by taking the infimum and the supremum over the set of EMMs admitted by an incomplete market:

$$\mathbf{C}^*(f_T)=\sup_{Q\in\mathscr{Q}}\mathbb{E}_Q[f_T], \qquad \mathbf{C}_*(f_T)=\inf_{Q\in\mathscr{Q}}\mathbb{E}_Q[f_T].$$

3 Pricing via market completions

In this section, we show how the upper and lower hedging prices can be represented in terms of market completions, or additional assets added to complete the original (B,S) market. In particular, in the market defined in Section 2.1, we characterize the set of completions that should be considered to obtain the no-arbitrage price bounds.

3.1 Market completions

Let S^c be an $(n-k)$-dimensional process representing the value of $(n-k)$ assets that will be added to the original market. We assume that the new assets have dynamics similar to the first k ones, that is, for $i \in \{k+1,\dots,n\}$,

$$dS_i(t) = S_i(t_-)\left(\nu_i(t)\,dt + \rho_i^V(t)\,dW(t) + \rho_i^J(t)\,dM(t)\right),$$

with $S_i(0) = s_0^i \in \mathbb{R}_+$, and where W and M are defined as in Section 2.1. The $(n-k)$-dimensional appreciation rate process ν and the matrix-valued processes ρ^V and ρ^J, with i^{th} row given by $\rho_i^V = (\rho_{i1}^V,\dots,\rho_{id}^V)$, and $\rho_i^J = (\rho_{i1}^J,\dots,\rho_{i(n-d)}^J)$, respectively, for $i = 1,\dots,n-k$, are predictable with respect to the filtration $\mathbb{F}$. As in the original market, we assume that ν and $\rho = [\rho^V\ \rho^J]$ are uniformly bounded in $(t,\omega) \in [0,T]\times\Omega$, and that $\rho_{il}^J \in [0,1]$, for $k+1 \le i \le n$ and $1 \le l \le n-d$.

We denote $\tilde{S} = (S,S^c)$, $\tilde{\mu} = (\mu,\nu)^\top$ and $\tilde{\sigma} = \binom{\sigma}{\rho}$, and only consider completions S^c such that the augmented market $(B,\tilde{S})$ is complete. That is, for a given market augmented with the completion S^c, we assume that $\det\tilde{\sigma} \neq 0$, and we define $\widetilde{\theta} = (\widetilde{\theta}^V,\widetilde{\theta}^J)$ by

$$\widetilde{\theta} = \widetilde{\sigma}^\top\left(\widetilde{\sigma}\widetilde{\sigma}^\top\right)^{-1}\widetilde{\mu}, \tag{8}$$

and $\widetilde{\gamma} = (\widetilde{\gamma}^V,\widetilde{\gamma}^J)$ by

$$\tilde{\gamma}^V = \tilde{\theta}^V \qquad\qquad \tilde{\gamma}^J = \lambda^{-1}\bullet\left(\lambda - \tilde{\theta}^J\right). \tag{9}$$

If $\widetilde{\theta}^J(t) < \lambda(t)$ for all $t \in [0,T]$, then $L_{\widetilde{\gamma}}$ is a true martingale and the $(B,\widetilde{S})$ market is complete.

Then, $Q_{\widetilde{\gamma}}$, defined by $\frac{dQ_{\tilde{\gamma}}}{dP} = L_{\tilde{\gamma}}$, is the unique EMM on the completed market $(B,\widetilde{S})$. It immediately follows that a contingent claim f_T has the unique no-arbitrage price $E[L_{\widetilde{\gamma}}f_T]$ in the completed market.

3.2 A special set of market completions

In this section, we want to express the upper and lower hedging prices defined in (6) and (7) in terms of market completions. To do so, we follow ideas similar to those used by [14] in a multivariate diffusion setting.

We let $\mathscr{R}_\rho$ be defined as the set of $(n-k)\times n$ matrix-valued $\mathbb{F}$-adapted process uniformly bounded in $(t,\omega) \in [0,T]\times\Omega$, with $\rho_{il}^J \in [0,1]$, for $k+1 \le i \le n$ and $1 \le l \le n-d$, such that $\det\binom{\sigma}{\rho} \neq 0$ P-a.s.

For each $\rho \in \mathscr{R}_\rho$, we denote by $\mathscr{D}_\rho$ the set of appreciation rate processes ν for which the associated market is complete. Therefore, we have

$$\mathscr{D}_\rho := \{\nu : \nu \text{ is } \mathbb{F}\text{-predictable, uniformly bounded, s.t. } \widetilde{\theta}^J(t) < \lambda(t)\ \forall\, t \in [0,T]\}.$$

For a given $\rho \in \mathscr{R}_\rho$, we define the *upper* and *lower completion prices* $\widetilde{\mathbf{C}}^*(f_T;\rho)$ and $\widetilde{\mathbf{C}}_*(f_T;\rho)$ by

$$\widetilde{\mathbf{C}}^*(f_T;\rho) = \sup_{\nu\in\mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(\nu,\rho)f_T], \qquad \widetilde{\mathbf{C}}_*(f_T;\rho) = \inf_{\nu\in\mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(\nu,\rho)f_T].$$

In the above, the density process $L_{\widetilde{\gamma}}(\nu,\rho) = L^V_{\widetilde{\gamma}}(\nu,\rho)L^J_{\widetilde{\gamma}}(\nu,\rho)$ defines the unique EMM on the market completed using assets with appreciation rate vector ν and diffusion coefficient matrix ρ.

As it is the case in the multidimensional diffusion market (see [14]), the upper and lower completion prices do not depend on the choice of ρ.

Proposition 3.1 *Fix ρ and $\rho' \in \mathscr{R}_\rho$. Then,*

$$\widetilde{\mathbf{C}}^*(f_T;\rho) = \widetilde{\mathbf{C}}^*(f_T;\rho'), \qquad\qquad \widetilde{\mathbf{C}}_*(f_T;\rho) = \widetilde{\mathbf{C}}_*(f_T;\rho').$$

Proof. The proof is very similar to the proof of Proposition 2.1 of [14]. Take the $(n-k)\times k$ and $(n-k)\times(n-k)$ predictable matrix valued processes C and D with $\det(D)\neq 0$ satisfying

$$\begin{pmatrix}\sigma\\ \rho'\end{pmatrix} = \begin{pmatrix} I & 0\\ C & D\end{pmatrix}\begin{pmatrix}\sigma\\ \rho\end{pmatrix},$$

where I denotes the identity matrix. Then it is possible to show that

$$\widetilde{\theta}_{\nu,\rho'} = \begin{pmatrix}\sigma\\ \rho'\end{pmatrix}^\top \left(\begin{pmatrix}\sigma\\ \rho'\end{pmatrix}\begin{pmatrix}\sigma\\ \rho'\end{pmatrix}^\top\right)^{-1}\begin{pmatrix}\mu\\ \nu\end{pmatrix} = \widetilde{\theta}_{\nu',\rho},$$

with $\nu' = D^{-1}(\nu - C\mu) \in \mathscr{D}_\rho$. The result follows. □

Since the upper and lower completion prices are independent of ρ, it is natural to only consider the market completions associated with a particular matrix-valued process ρ. Henceforth, we fix $\bar\rho \in \mathscr{R}_\rho$ satisfying

$$\sigma\bar\rho^\top = 0 \qquad \text{and} \qquad \bar\rho\bar\rho^\top = I. \tag{10}$$

It follows that $\widetilde{\theta}_{\nu,\bar\rho} = \theta + \vartheta_\nu$ with $\vartheta_\nu = \bar\rho^\top(\bar\rho\bar\rho^\top)^{-1}\nu$ for any $\nu \in \mathscr{D}_{\bar\rho}$.

The density process $L_{\widetilde{\gamma}}(\nu,\bar\rho)$ of the EMM associated with each market completion with parameters $\bar\rho$ and $\nu \in \mathscr{D}_{\bar\rho}$ can then be written as

$$\begin{aligned} L_{\widetilde{\gamma}}(t;\nu,\bar\rho) = & e^{-\int_0^t \gamma^V(s)^\top dW(s) - \int_0^t \vartheta^V_\nu(s)^\top dW(s) - \frac{1}{2}\int_0^t \|\theta^V(s)+\vartheta^V_\nu(s)\|^2 ds} \\ & \times e^{-\int_0^t \theta^J(s)\, ds - \int_0^t \vartheta^J_\nu(s)\, ds} \prod_{l=1}^{n-d}\prod_{s\le t}\lambda^{-1}(s)\bullet\left(\lambda(s) - \theta^J(s) - \vartheta^J_\nu(s)\right)\Delta N_l(s). \end{aligned}$$

In the above, $\theta = (\theta^V, \theta^J)$ is as defined in (2), and is therefore independent of the market completion.

3.3 Completion price bounds

Finally, we highlight the equivalence between the set of EMMs $\mathscr{Q}$ of the original market and the set of market completions associated with the fixed matrix process ρ satisfying (10).

Lemma 1. *Fix $\rho \in \mathscr{R}_\rho$. Then for any $\gamma \in \Gamma$, it is possible to find $\nu \in \mathscr{D}_\rho$ such that*

$$\gamma^V = \tilde{\theta}^V_{\nu,\rho}, \qquad \gamma^J = \lambda^{-1} \bullet (\lambda - \widetilde{\theta}^J_{\nu,\rho}).$$

Proof. To find such a $\nu \in \mathscr{D}_\rho$, it suffices to let $\widetilde{\theta}^V = \gamma^V$ and $\widetilde{\theta}^J = \lambda - \lambda \bullet \gamma^J$. Then, since $\widetilde{\sigma}^\top(\widetilde{\sigma}\widetilde{\sigma}^\top)^{-1}$ is an $n \times n$ matrix, there is only one solution $\widetilde{\mu} = (\mu, \nu)$ that satisfies

$$\widetilde{\theta} = \widetilde{\sigma}^\top(\widetilde{\sigma}\widetilde{\sigma}^\top)^{-1}\widetilde{\mu}. \tag{11}$$

Indeed, this solution is given by $\widetilde{\mu} = \widetilde{\sigma}\widetilde{\theta}$, and $\nu \in \mathscr{D}_\rho$ by definition. □

It follows from Lemma 1 that any measure $Q \in \mathscr{Q}$ can be recovered by completing the market using a market completion S^c with parameters $\bar{\rho}$ and ν, for some $\nu \in \mathscr{D}_{\bar{\rho}}$.

We can also show that any $\nu \in \mathscr{D}_{\bar{\rho}}$ defines an element $\gamma \in \Gamma$.

Lemma 2. *Fix $\rho \in \mathscr{R}_\rho$, with ρ satisfying* (10). *Then for any $\nu \in \mathscr{D}_\rho$, the resulting $\widetilde{\gamma}$, as defined by* (8) *and* (9) *is an element of Γ.*

Proof. From (10), we have $\widetilde{\theta} = \theta + \vartheta$, and the resulting $\tilde{\gamma}$ solves (4), since $\sigma\theta^\top = \mu$ and $\sigma\vartheta^\top = 0$. □

Therefore, the unique EMM resulting from any market completion S^c with parameters $\bar{\rho}$ and ν, with $\nu \in \mathscr{D}_{\bar{\rho}}$ is an element of $\mathscr{Q}$.

Lemmas 1 and 2 confirm that the set of market completions with diffusion coefficient matrix ρ fixed spans the set of EMMs $\mathscr{Q}$ on the complete market. It is therefore possible to express the range of no-arbitrage prices for a contingent claim f_T in terms of the set of market completions.

Proposition 3.2 *The upper and lower hedging prices $\mathbf{C}^*(f_T)$ and $\mathbf{C}_*(f_T)$ coincide with the upper and lower completion prices $\tilde{\mathbf{C}}^*(f_T)$ and $\tilde{\mathbf{C}}_*(f_T)$, and we have*

$$\mathbf{C}^*(f_T) = \sup_{\nu \in \mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(T;\nu,\rho) f_T], \qquad \mathbf{C}_*(f_T) = \inf_{\nu \in \mathscr{D}_\rho} E[L_{\widetilde{\gamma}}(T;\nu,\rho) f_T].$$

4 Concluding remarks

In a pure diffusion market, the process $L_{\widetilde{\gamma}}(\nu,\rho)$ can be expressed as the product of two (local) martingales; one pertaining to the original market, and the other one associated with the market completion. This makes market completion techniques very useful in the context of hedging and portfolio optimization problems. The addition of jumps in the market generally removes the possibility of such an expression, as is remarked at the end of Section 1 of [13].

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