Institut canadien des dérivés
Canadian Derivatives Institute

L'Institut bénéficie du soutien financier de l'Autorité des marchés financiers ainsi que du ministère des Finances du Québec

**Document de recherche**

**DR 14-02**

# A Random Field LIBOR Market Model

Novembre 2014

Ce document de recherche a été rédigée par :

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# A Random Field LIBOR Market Model

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January 1, 2014

A random field LIBOR market model (RFLMM) is proposed by extending the LIBOR market model, with interest rate uncertainties modeled via a random field. First, closed-form formulas for pricing caplet and swaption are derived. Then the random field LIBOR market model is integrated with the lognormal-mixture model to capture the implied volatility skew/smile. Finally, the model is calibrated to cap volatility surface and swaption volatilities. Numerical results show that the random field LIBOR market model can potentially outperform the LIBOR market model in capturing caplet volatility smile and the pricing of swaptions, in addition to possessing other advantages documented in the previous literature (no need of frequent re-calibration or to specify the number of factors in advance).

## 1 Introduction

In this paper we extend the LIBOR market model (LMM) by describing forward rate uncertainties as a random field. A researcher does not need to specify the number of factors in advance or to frequently recalibrate random field models. The LIBOR market model (see Brace, Gatarek, and Musiela (BGM)[7], Jamshidian[23], and Miltersen, Sandmann, and Sondermann [32]) is based on the assumption that each forward LIBOR rate follows a driftless Brownian motion under its own forward measure, which justifies the use of Black's formula for the pricing of interest rate options, such as caplets and swaptions. It is very fast to calibrate the LMM to market data. As a result, the LMM has become very popular among practitioners for interest rate modeling and derivatives pricing.

However, the LMM has several limitations. Firstly, like any finite factor interest rate models, the LMM requires the specification of the number of

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factors in advance. However, the number of factors required is typically not obvious to the modeler. Longstaff et. al. [30] shows that using even the best-fitting single-factor model can be suboptimal if the term structure is driven by multi-factors. Secondly, frequent re-calibration is needed for the LMM. Like other no-arbitrage models that are consistent with the current term structure at the current point in time, the LMM requires re-calibration of the parameters in order to fit the new term structure the following day (or week). Finally, the original LMM only generates flat implied volatility curve of caplets and swaptions and thus cannot capture the implied volatility smile as observed in these markets. Several extensions have been proposed to capture the smile/skew of implied volatilities. The first approach is the so called local volatility models. For example, the constant elasticity of variance (CEV) model by Andersen and Andreasen [2] and the displaced diffusion (DD) model by Joshi and Rebonato [24] generate a monotonic skew, but not smile, of implied volatility. Another approach is a stochastic volatility model, in which volatility itself is modeled as a stochastic process. For example, Andersen and Brotherton-Ractliffe [4], Wu and Zhang [45] produce additional curvature to the volatility curve. Hagan et al.[14] propose a stochastic volatility extension of the CEV model, termed the SABR (Stochastic Alpha-Beta-Rho) model to capture the smile or skew of caps and swaptions.

A new methodology that models interest rate innovations by a random field overcomes the first two of the above limitations. Kennedy [25, 26] and Goldstein [13] first introduce this methodology. Pang [35], Longstaff et al.[30] and Bester [5] investigate random field models from different perspectives such as model calibration and option pricing.

In this paper, we aim to draw advantages from the random field modeling approach while retaining the tractability of the LMM. In particular, we introduce an extended LIBOR market model where forward rate uncertainties are driven by a random field. We therefore name it the random field LIBOR market model (RFLMM). Then the RFLMM is integrated with the lognormal-mixture local volatility model to capture implied volatility skews/smiles. It will be illustrated that this model can generate the implied volatility smile in interest rate caps and is potentially more accurate in calibration. We also demonstrate through our implementation that it is unnecessary to choose the number of factors in advance. It is worth noting that the primary contribution of this paper is theoretical, rather than empirical, although we do offer a "smile" model extension and a calibration example for illustrative purposes. Empirical estimation using extensive historical data is conducted in a companion paper.

The rest of the paper is organized as follows. Section 2 extends the LIBOR market model to the random field setting. Formulas for caplets and swaptions pricing are provided. Section 3 integrates the model with lognormal-mixture local volatility model. Section 4 calibrates the model to market data. Section 5 concludes the paper.

# 2 A Random Field LIBOR Market Model

In this section we derive an extended LIBOR market model with uncertainties described by a random field. The extended model is therefore referred to as the random field LIBOR market model (RFLMM). Firstly, we introduce random field as a description of uncertainty in Sec.2.1. Secondly, we review the advantages of modeling interest rates as a random field in Sec.2.2. Thirdly, we derive the random field LIBOR market model in Sec.2.3, and closed-form formulas for pricing European caplets and swaptions in Sec.2.4.

## 2.1 Random Field

A random field is a stochastic process that is indexed by a spatial variable, as well as a time variable. For example, if we would like to measure the temperature at position $u$ and time $t$ with $u \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, the measure can be modeled as a random variable $W(t,u)$. The collection of

$$\{W(t,u) : (t,u) \in \mathbb{R}^+ \otimes \mathbb{R}^n\}$$

is a random field.

A random field is generalization of a Brownian motion. For fixed maturity $T$, the random field $W(t,T)$ is a Brownian motion, i.e. $dW(t,T) := [dW(t+dt,T) - dW(t,T)]$ follows the normal distribution $\mathcal{N}(0,dt)$. In addition, the correlation between $dW(t,T_1)$ and $dW(t,T_2)$ for any $t < T_1 \leq T_2$ is specified by

$$c(u,v) = \mathrm{Corr}[dW(t,u), dW(t,v)], \tag{1}$$

with $\lim_{u\to v} c(u,v) = 1$, where the differential notation $d$ is used to denote the increments in the $t$-direction. Given a specification of the correlation structure $c(u,v)$, Bester [5] describes the construction of random field as follows. Assuming that for a correlation function $c(u,v)$, there exists a symmetric function $g(u,v)$ such that

$$c(u,v) = \int_0^\infty g(u,z)g(v,z)dz,$$

with $\int_0^\infty |g(u,z)|^2 dz = 1$. The random field $W(t,u)$ can be defined as

$$W(t,u) = W(0,0) + \int_0^\infty \int_0^t g(u,z)\epsilon(s,z)dsdz, \tag{2}$$

or equivalently

$$dW(t,u) = [\int_0^\infty g(u,z)\epsilon(t,z)dz]dt \tag{3}$$

From Eq.(3) we can see that the increments of a random field are weighted average of white noise at time $t$. Here $g(u, z)$ is the weight of $\epsilon(t, z)$ at location $z$ in determining the change of the field at location $u$. A formal theoretic definition of random field and other treatments can be found in Adler [1], Gikhman-Skorokhod [12] and Khoshnevisan [28].

### 2.2 Interest Rate Modeling in the Random Field Setting

Modeling interest rates as a random field was introduced by Kennedy [25, 26] and Goldstein [13]. Limiting the scope to Gaussian random field, Kennedy [25] obtains the form of the drift terms of the instantaneous forward rates processes necessary to preclude arbitrage under the risk neutral measure. Goldstein [13] extends the work to the case of non-Gaussian random field. We first review the Kennedy-Goldstein Framework.

**The Kennedy-Goldstein Framework.** Define zero coupon bond price

$$P(t,T) = e^{-\int_t^T f(t,u)du}, \tag{4}$$

where $f(t, u)$ is the instantaneous forward rate and $t$, $T$ are current time and maturity respectively. According to Goldstein [13], if we require the discounted bond price

$$e^{-\int_0^t r(s)ds} P(t,T), \tag{5}$$

where $r(t)$ is the instantaneous spot rate, to be a martingale under the risk neutral measure $\mathbb{Q}$, the drift term of the forward rate under $\mathbb{Q}$ must be given by $\sigma(t,T) \displaystyle\int_t^T \sigma(t,u)c(t,T,u)du$, to rule out arbitrage. Thus the dynamics of the instantaneous forward rates $f(t, T)$ under the risk neutral measure is given by

$$df(t,T) = \sigma(t,T) \int_t^T \sigma(s,u)c(s,T,u)duds + \sigma(t,T)d\widetilde{W}(t,T), \tag{6}$$

with the correlation structure

$$\mathrm{Corr}[dW(t,T_1), dW(t,T_2)] = c(t,T_1,T_2), \tag{7}$$

where $\lim_{\Delta T \to 0} c(t, T, T + \Delta T) = 1$ and $\widetilde{W}(t, T)$ is a random field under the risk neutral measure $\mathbb{Q}$.

Goldstein [13] and Pang [35] argue that modeling the term structure of interest rates as a random field provides many advantages. First, it is unnecessary to determine the number of factors prior to calibration or estimation. Random field models are infinite-factor models, since the instantaneous forward rates in random field models form a continuum. In fact random field models accommodate both finite and infinite-factor models and thus all finite-factor models are special cases of random field models. It can be shown that the Gaussian random fields can be interpreted as a linear combination of infinite number of Brownian motions indexed by different forward rate maturities, $T$, under mild technical conditions. Pang [35] shows that it is possible to reduce Eq.(6) to a $d$-factor HJM model

$$df(t,T) = \boldsymbol{\sigma}(t,T) \cdot \int_t^T \boldsymbol{\sigma}(t,u)dudt + \boldsymbol{\sigma}(t,T) \cdot d\widetilde{\mathbf{W}}(t), \tag{8}$$

by taking

$$d\widetilde{W}(t,T) = \frac{1}{\sigma(t,T)} \sum_{i=1}^{d} \sigma_i(t,T) d\widetilde{W}_i(t), \tag{9}$$

where $\sigma(t,u)$ is a $d-$dimension vector and $\cdot$ is the inner product of two vectors. In fact, for the random field model, the covariance of instantaneous changes of forward rates is given as

$$\mathrm{Cov}[df(t,u), df(t,v)] = \sigma(t,u)\sigma(t,v)c(u,v). \tag{10}$$

While in the HJM framework, we have

$$\mathrm{Cov}[df(t,u), df(t,v)] = \boldsymbol{\sigma}(t,u) \cdot \boldsymbol{\sigma}(t,v). \tag{11}$$

If $c(u,v)$ can be written as

$$c(u,v) = \mathbf{e}(t,u) \cdot \mathbf{e}(t,v), \tag{12}$$

with $|\mathbf{e}(t,T)| = 1$, we can take

$$\boldsymbol{\sigma}(t,T) = \sigma(t,T)\mathbf{e}(t,T). \tag{13}$$

Thus the Gaussian field model Eq.(6) is reduced to HJM model Eq.(8). This suggests that random field models can be viewed as an infinite-factor generalization of HJM models. This shows the difference when calibrating HJM and random field models. We can directly specify the correlation structure without specifying the number of factors.

Second, it is unnecessary to frequently recalibrate random field models. As pointed out in Buraschi and Corielli [10],the finite-factor models completely determines the future trend of the yield curve and thus its possible shapes in the future. For instance, the $d$-factor models can at most fit $d$ points on the yield curve and the volatility curve on a specific date. On the next day, without re-calibration, the same specification will typically miss those points. In practice, the most common solution is to frequently re-calibrate the model by inputting a new term structure. However, as Pang [35] points out, the re-initialization of yield curve on each new date would violate the no-arbitrage principle. In other words, the re-calibration violates the self-financing condition of the replication strategy since it implies a change in the conditional distribution of the process with respect to which the replicating portfolio weights are computed. In $d$-factor HJM term structure models, any security can be perfectly hedged by a preferred choice of $d$ assets. However, in a random field model, the innovation of each instantaneous forward is imperfectly correlated with that of any linear combination of other instantaneous forwards. Thus random field models have enough degree of freedom to fit the current yield curve. More discussion of re-calibration can be found in Pang [35], which demonstrate that the calibration of a random field model permits more stability over time and frequent re-calibration can be avoided, in contrast to a $d$-factor HJM model. Pang [35] examines the stability of the covariance function and show that the function maintains similar shapes throughout a long period of time (at least one month). The eigenvalues and corresponding eigenfunctions of the implied zero rate covariance matrix remain mostly the same during the sample period. In contrast, when Pang applies principal component analysis (PCA) to extract the eigenvalues and eigenvectors to examine the HJM models, he discovers that the eigenvalues and eigenvectors in HJM models are very unstable over time.

### 2.3 A Random Field Libor Market Model (RFLMM)

In this section, we derive the dynamics of LIBOR rates $L_k(t)$ with uncertainty terms modeled as a random field, under the risk neutral measure $\mathbb{Q}$, and the $T_j$-forward measure $\mathbb{Q}^{T_j}$, for $j = 0, 1, ..., N$.

Let us consider the time structures $\{T_0, T_1, ..., T_N\}$ with time intervals $\delta_k = T_k - T_{k-1}, k = 1, ..., N$. For $t < T_{k-1} < T_k$, the LIBOR forward rate $L_k(t)$ is defined as

$$L_k(t) = \frac{1}{\delta_k}[\frac{P(t, T_{k-1})}{P(t, T_k)} - 1], \tag{14}$$

and the zero coupon bond price $P(t, T)$ is defined in Eq.(4). Given the dynamics of $f(t, T)$ in Eq.(6), by Itô's formula we can derive that the dynamics of the zero coupon bond price $P(t, T)$ in the random field setting is

$$dP(t, T) = P(t, T)[r(t)dt - \int_t^T \sigma(t, u)d\widetilde{W}(t, u)du]. \tag{15}$$

The dynamics of $L_k(t)$ is determined by those of zero coupon bonds. By Itô's formula we can derive the dynamics of $L_k(t)$ under the risk neutral measure $\mathbb{Q}$:

$$\begin{aligned} dL_k(t) \quad = \quad & \frac{1}{\delta_k}\frac{P(t, T_{k-1})}{P(t, T_k)}[\int_{T_{k-1}}^{T_k} \sigma(t, u)d\widetilde{W}(t, u)du + \int_t^{T_k} \sigma(t, u)d\widetilde{W}(t, u)du \\ & \int_{T_{k-1}}^{T_k} \sigma(t, u)d\widetilde{W}(t, u)du]. \end{aligned} \tag{16}$$

Now let us derive the dynamics of the forward rates under the $T_k$-forward measure. Suppose that there exists a function $\theta(t, T_k, u)$ such that $dW^{T_k}(t, u) :=$

$\theta(t,T_k,u)dt+d\widetilde{W}(t,u)$ has normal distribution $\Phi(0,dt)$ under the $T_k$-forward measure. Using the fact that if $L_k(t)$ is a martingale under the $T_k$-forward measure then the drift term should vanish, we can conclude that

$$\theta(t,T_k,u) = \int_t^{T_k} \sigma(t,v)c(u,v)dv. \tag{17}$$

In the rest of this paper, we assume that volatility is of the form $\sigma(t,T) = \sigma e^{-\kappa(T-t)}$, and the correlation is of the form $corr[dW^{T_k}(t,u), dW^{T_k}(t,v)] = e^{-\rho|u-v|}$. Thus we have the dynamics of $L_k(t)$ under the $T_k$-forward measure as shown in the following proposition and corollaries.

**Proposition 2.1. (Forward LIBOR dynamics under the associated forward measures)** *The dynamics of $L_k(t)$ under the $T_k$-forward measure is described by the following equation*

$$dL_k(t) = \frac{1}{\delta_k}\frac{P(t,T_{k-1})}{P(t,T_k)}\int_{T_{k-1}}^{T_k} \sigma(t,u)dW^{T_k}(t,u)du, \tag{18}$$

*where $W^{T_k}(t,u)$ is a random field under the $T_k$-forward measure.*

See Appendix 1 for the proof of Proposition 2.1.

**Corollary 2.2.** *If $d\widetilde{W}(t,u)$ is normally distributed according to $\mathcal{N}(0,dt)$ under the risk neutral measure, then*

$$dW^{T_k}(t,u) := \int_t^{T_k} \sigma(t,v)c(u,v)dvdt + d\widetilde{W}(t,u)$$

*is normally distributed according to $\mathcal{N}(0,dt)$ under the $T_k$-forward measure (using $P(t,T_k)$ as a numeraire).*

**Corollary 2.3.** *If $corr[dW^{T_k}(t,u), dW^{T_k}(t,v)] = c(u,v)$, then $\int_{T_{k-1}}^{T_k} \sigma(t,u)dW^{T_k}(t,u)du$ is normally distributed with mean 0 and variance $\int_{T_{k-1}}^{T_k}\int_{T_{k-1}}^{T_k} \sigma(t,x)\sigma(t,y)c(x,y)dxdydt$ under the $T_k$-forward measure.*

Proposition 2.1 shows that $L_k(t)$ is log-normally distributed under the $T_k$-forward measure. Corollary 2.2 describes the dynamics of the random field under the $T_k$-forward measure and Corollary 2.3 provides the distribution of the integral of the random field, given the correlation structure $c(u,v)$. These results are essential for the derivation of the RFLMM.

Now we can derive the dynamics of forward rates $L_k(t)$ under the $T_j$-forward measure, $j = 1,2,...,N$. From Eq.(18) and Corollary 2.2, we can obtain the relation of $dW^{T_k}(t,u)$ and $dW^{T_{k+1}}(t,u)$ as follows,

$$dW^{T_k}(t,u) = dW^{T_j}(t,u) - \sum_{i=k+1}^{j}\int_{T_{i-1}}^{T_i} \sigma(t,v)c(u,v)dvdt, \tag{19}$$

for $j > k$. As we know, $L_k(t)$ is a martingale under the $T_k$-forward measure. By the martingale representation theorem, there exists a function $\zeta_k(t,u)$, such that

$$dL_k(t) = \int_{T_{k-1}}^{T_k} \zeta_k(t,u) dW^{T_k}(t,u) du. \tag{20}$$

Comparing the above equation with Eq.(18), we can simply take

$$\zeta_k(t,u) = \xi_k(t,u) L_k(t); \tag{21}$$

$$\xi_k(t,u) = \frac{\delta_k L_k(t) + 1}{\delta_k L_k(t)} \sigma(t,u), \tag{22}$$

and the dynamics of $L_k(s)$ under the $T_k$-forward measure is given by

$$dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u) dW^{T_k}(t,u) du. \tag{23}$$

The derivation in cases where $j < k$ are similar. Thus we have the following proposition.

**Proposition 2.4. (Forward LIBOR dynamics under other forward measures)** *The dynamics of $L_k(t)$ under the $T_j$-forward measure, in three cases $j < k, j = k, j > k$, are described respectively by the following equations*

$$\begin{cases} dL_k(t) = L_k(t) \displaystyle\int_{T_{k-1}}^{T_k} \xi_k(t,u)[dW^{T_j}(t,u) + \Lambda_j^k(t,u)dt]du, & j < k; \\ dL_k(t) = L_k(t) \displaystyle\int_{T_{k-1}}^{T_k} \xi_k(t,u) dW^{T_j}(t,u) du, & j = k; \\ dL_k(t) = L_k(t) \displaystyle\int_{T_{k-1}}^{T_k} \xi_k(t,u)[dW^{T_j}(t,u) - \Lambda_k^j(t,u)dt]du, & j > k. \end{cases} \tag{24}$$

*with*

$$\Lambda_j^k(t,u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t,v) c(u,v)}{\delta_i L_i(t) + 1} dv,$$

*where $W^{T_j}(t,u)$ is a random field under the $T_j$-forward measure. The above equations admit a unique solution if the coefficient $\xi_k(\cdot,\cdot)$ are locally bounded, locally Lipschitz continuous and predictable.*

See Appendix 2 for the proof of Proposition 2.4.

**Proposition 2.5. (Consistency between the LMM and the RFLMM)** *The HJM framework is a special case of the Kennedy-Goldstein framework, since Eq.(24) in Proposition 2.4 reduces to Eq.(6), if we take $\delta_i \to 0$. The LMM is a special case of the RFLMM, since Eq.(24) in Proposition 2.4* *reduces to Eq.(25), if we take the value of $dW^T(t,u)$ on $[T_{k-1}, T_k]$ to be $dW_k^T(t)$.*

$$\begin{cases} dL_k(t) = L_k(t)\xi_k(t)[dW_k^{T_j}(t) + \sum_{i=j+1}^{k} \frac{\delta_i \rho_{i,k}(t) L_i(t) \xi_i(t)}{\delta_i L_i(t) + 1} dt], & j < k; \\ dL_k(t) = L_k(t)\xi_k(t) dW_k^{T_j}(t), & j = k; \\ dL_k(t) = L_k(t)\xi_k(t)[dW_k^{T_j}(t) - \sum_{i=k+1}^{j} \frac{\delta_i \rho_{i,k}(t) L_i(t) \xi_i(t)}{\delta_i L_i(t) + 1} dt], & j > k. \end{cases} \quad (25)$$

See Appendix 3 for the proof of Proposition 2.5.

Proposition 2.5 shows that the LIBOR market model Eq.(25) is a special case of the random field LIBOR market model.

## 2.4 Option Pricing in the Random Field LIBOR Market Model

In this section we derive the Black-Scholes equation for the pricing of derivatives in the random field case and the closed-form formulas for caplets and swaptions. Similar to the Black-Scholes equation for the price of derivatives in the Brownian motion case, we derive a Black-Scholes type equation in the random field case in the following proposition.

**Proposition 2.6. (The Black-Scholes equation in the random field setting for time dependent parameters)** *Suppose that we have an option on some underlying asset $S$, which follows dynamics*

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \int_{t_1}^{t_2} \xi(t,u) dW(t,u) du, \quad (26)$$

*with time dependent parameters. Then the price of derivatives $V$ satisfies the equation:*

$$\frac{\partial V}{\partial t} + \frac{1}{2}[\int_{t_1}^{t_2} \int_{t_1}^{t_2} \xi(t,u)\xi(t,v)c(u,v)dudv] \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - r(t)V = 0. \quad (27)$$

*We denote it as the Black-Scholes equation for option pricing with random field. For a call option with strike $K$ and maturity $T$, the price is given by*

$$C = S\Phi(\tilde{d}_1) - Ke^{-\int_0^T r(u)du} \Phi(\tilde{d}_2),$$

*where*

$$\begin{aligned} \tilde{d}_1 &= \frac{1}{\sqrt{\int_0^T \widehat{\xi}^2(u,t_1,t_2)du}} [\ln \frac{S}{K} + \int_0^T (r(u) + \frac{\widehat{\xi}^2(u,t_1,t_2)}{2})du]; \\ \tilde{d}_2 &= \tilde{d}_1 - \sqrt{\int_0^T \widehat{\xi}^2(u,t_1,t_2)du}; \\ \widehat{\xi}^2(t,t_1,t_2) &= \int_{t_1}^{t_2} \int_{t_1}^{t_2} \xi(t,u)\xi(t,v)c(u,v)dudv, \end{aligned}$$

*and $\Phi(d)$ is the normal cumulative distribution function.*

See Appendix 4 for the proof of Proposition 2.6. We can get a Black-type formula for pricing caplets and swaptions from Proposition 2.6.

### 2.4.1 Random field LMM formula for caplets

The payoff of a caplet at time $T_k$ is

$$\delta_k[L_k(T_{k-1}) - K]^+.$$

The time $t$ price of a caplet is given by

$$\mathbf{Cplt}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k)\mathbb{E}^{T_k}[[L_k(T_{k-1}) - K]^+|\mathcal{F}_t].$$

Suppose that $L_k(t)$ follows dynamics given by Eq.(23) under the $T_k$-forward measure. If we assume that $\xi_k(t, u)$ is deterministic, then by Corollary 2.3, $L_k(t)$ is log-normally distributed and by Black's formula the time $t$ price is

$$\mathbf{Cplt}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k)\mathbf{Black}(K, L_k(t), \sigma_k^{Black,RF}\sqrt{T_{k-1} - t}),$$

where

$$\mathbf{Black}(K, L_k(t), \sigma_k^{Black,RF}\sqrt{T_{k-1} - t}) = L_k(t)\Phi(d_1) - KN\Phi(d_2), \tag{28}$$

$$\begin{aligned} d_1 &= \frac{\log(L_k(t)/K) + (\sigma_k^{Black,RF})^2(T_{k-1} - t)/2}{\sigma_k^{Black,RF}\sqrt{T_{k-1} - t}}, \\ d_2 &= \frac{\log(L_k(t)/K) - (\sigma_k^{Black,RF})^2(T_{k-1} - t)/2}{\sigma_k^{Black,RF}\sqrt{T_{k-1} - t}} = d_1 - \sigma_k^{Black,RF}\sqrt{T_{k-1} - t}, \end{aligned}$$

and

$$\sigma_k^{Black,RF} = \sqrt{\frac{1}{T_{k-1} - t}\int_t^{T_{k-1}}[\int_{T_{k-1}}^{T_k}\int_{T_{k-1}}^{T_k}\xi_k(s, x)\xi_k(s, y)c(x, y)dxdy]ds}. \tag{29}$$

We can specify the volatility and correlation structure to get particular forms of the pricing formula.

### 2.4.2 Random field LMM formula for swaptions

The payoff of a European swaption at time $T_k$ is

$$\delta_k[S_{i,j}(T_{k-1}) - K]^+.$$

The time $t$ price is therefore

$$[S_{i,j}(t) - K]^+\sum_{k=i+1}^{j}\delta_k P(t, T_k),$$

where the swap rate

$$S_{i,j}(t) = \frac{\sum_{k=i+1}^{j} \delta_k P(t,T_k) L_k(t)}{\sum_{k=i+1}^{j} \delta_k P(t,T_k)} = \frac{P(t,T_i) - P(t,T_j)}{\sum_{k=i+1}^{j} \delta_k P(t,T_k)}, \tag{30}$$

for $0 \le t \le T_i$, $i < j \le N$.

In the swap market model, the swap rates are assumed to follow dynamics

$$dS_{i,j}(t) = S_{i,j}(t)\eta_{i,j}(t)dW^{i,j}(t). \tag{31}$$

The time $t$ price of a swaption can be computed from the Black's formula as

$$\textbf{Swaption}(t, K, T_i, T_j) = A\ \textbf{Black}(K, S_{i,j}(t), \overline{\sigma}_{i,j}^{Black}\sqrt{T_{k-1} - t}), \tag{32}$$

where $\overline{\sigma}_{i,j}^{Black} := \sqrt{\frac{1}{T_{k-1} - t}\int_t^{T_{k-1}} \| \eta_{i,j}(s) \|^2\, ds}$, $A = \sum_{k=i+1}^{j} \delta_k P(t,T_k)$.

However, the assumption of log-normally distributed forward LIBOR and log-normally distributed swap rates can not hold simultaneously. As a result, swaptions cannot be priced directly using Black's formula within the LIBOR market model. To price the swaptions within the LIBOR market model, we need to rewrite the implied volatility of the swaption in terms of forward LIBOR $L_k(t)$, which has log-normally distribution under LIBOR market model. Given

$$\frac{P(t,T_k)}{P(t,T_i)} = \prod_{j=i+1}^{k} \frac{1}{1 + \delta_j L_j(t)},$$

for $k \ge i+1$, by dividing through $P(t,T_j)$, the swap rate defined in Eq.(30) can be written as

$$S_{i,j}(t) = \frac{\prod_{l=i+1}^{j}(1 + \delta_l L_l(t)) - 1}{\sum_{k=i+1}^{j} \delta_k \prod_{l=k+1}^{j}(1 + \delta_l L_l(t))},$$

or equivalently

$$\ln S_{i,j}(t) = \ln\{\prod_{l=i+1}^{j}(1 + \delta_l L_l(t)) - 1\} - \ln\{\sum_{k=i+1}^{j} \delta_k \prod_{l=k+1}^{j}(1 + \delta_l L_l(t))\},$$

for $j - 1 \ge k \ge i + 1$, where $\prod_m^n = 1$ if $m > n$. According to Hull and White[22] and from Itô's formula, the uncertainty term of swap rates in the LMM model can be written as:

$$\sum_{k=i+1}^{j} \frac{1}{S_{i,j(t)}} \frac{\partial S_{i,j}(t)}{\partial L_k(t)} dW_k^{T_k}(t) = \sum_{k=i+1}^{j} \frac{\delta_k L_k(t) \gamma_k^{i,j}(t)}{1 + \delta_k L_k(t)} \xi_k(t) dW_k^{T_k}(t),$$

where

$$\gamma_k^{i,j}(t) = \frac{\prod_{l=i+1}^{j}(1+\delta_l L_l(t))}{\prod_{l=i+1}^{j}(1+\delta_l L_l(t)) - 1} - \frac{\sum_{m=i+1}^{k-1}\delta_m \prod_{l=m+1}^{j}(1+\delta_l L_l(t))}{\sum_{m=i+1}^{j}\delta_m \prod_{l=m+1}^{j}(1+\delta_l L_l(t))}.$$

Using Black's formula the time $t$ price of swaption is

$$\mathbf{Swaption}(t, K, T_i, T_j) = A\ \mathbf{Black}(K, S_{i,j}(t), \overline{\sigma}_{i,j}^{Black,RF}\sqrt{T_{k-1}-t}),$$

where

$$\begin{aligned}
\overline{\sigma}_{i,j}^{Black,RF} &= \sqrt{\frac{1}{T_i - t}\int_t^{T_i} \| \sum_{k=i+1}^{j} \frac{\delta_k L_k(s)\gamma_k^{i,j}(s)}{1+\delta_k L_k(s)} \int_{T_{k-1}}^{T_k} \xi_k(s) dW(s,u)\|^2 ds} \\
&= \sqrt{\frac{1}{T_i - t}\sum_{k=i+1}^{j}\sum_{l=i+1}^{j} \frac{\delta_k L_k(t)\gamma_k^{i,j}(t)}{1+\delta_k L_k(t)} \frac{\delta_l L_l(t)\gamma_l^{i,j}(t)}{1+\delta_l L_{l+1}(t)}} \times \\
&\quad \sqrt{\int_t^{T_i}\int_{T_{k-1}}^{T_k}\int_{T_{l-1}}^{T_l} \xi_k(s,x)\xi_l(s,y)c(x,y)dxdyds.} \qquad (33)
\end{aligned}$$

The last equation is obtained by using standard freezing approximation techniques, i.e. approximatively evaluating the LIBOR rates $L_k(s)$, $t \le s \le T_i$ appearing in the instantaneous volatility at initial time $t$.

**Remark 2.7.** Standard Freezing Approximation Techniques.
*The last equation is obtained by approximatively evaluating the LIBOR rate $L_k(s)$, $t \le s \le T_i$ at initial time $t$. This approximation technique is highly accurate according to Hull and White [21]. Hull and White [21] compared the prices of swaptions calculated by the approximation formula above with the price calculated from a Monte Carlo simulation and found the two to be very close. We will use this approximation in several places in this paper.*

In this section we have derived the random field LIBOR market model (RFLMM) (Eq.(24)) and formulas for the implied volatility for caplets (Eq.(29)) and swaptions (Eq.(33)). These two formulas can be used to calibrate the RFLMM.

# 3 Random Field Lognormal-mixture Volatility Smile Model

In this section, we derive a random field lognormal-mixture volatility smile model. First, we review implied volatility smiles in interest rates options in Sec.3.1. Then we extend the lognormal-mixture model to the random field case in Sec.3.2.

### 3.1 Volatility Smile

It is well known that the lognormal LMM has the main drawback of producing constant implied volatility for any given maturity. From the Black's formula for caplet price, we can see that the volatility of the forward rate does not depend on the option strike $K$. However, each caplet market price requires its own Black volatility depending on the caplet strike $K$. In other terms, there is not a single volatility $\sigma_k$ such that both

$$\mathbf{Cplt}(t, K_1, T_{k-1}, T_k) = \delta_k P(t, T_k)\mathbf{Black}(K_1, L_k(t), \sigma_k\sqrt{T_{k-1} - t})$$

and

$$\mathbf{Cplt}(t, K_2, T_{k-1}, T_k) = \delta_k P(t, T_k)\mathbf{Black}(K_2, L_k(t), \sigma_k\sqrt{T_{k-1} - t})$$

hold. The market observation shows that we need two different volatilities $\sigma_k(K_1)$ and $\sigma_k(K_2)$ to use Black's formula to match market prices. The volatility smile or skew of the $T_k$-expiry caplet is the curve $K \to \sigma_k(K)/\sqrt{T_k - t}$. The reason that the curve is called volatility 'smile' or 'skew' is that the curve usually displays 'smiley' or 'skewed' shapes.

The model's volatility smile is generated as follows. Given a starting strike $K$, we compute the caplet price using

$$\mathbf{Cplt}(K) = \delta_k P(t, T)E^{T_k}[L_k(T_{k-1}) - K)^+|\mathcal{F}_t],$$

with $L_k(t)$ following dynamics Eq.(34). Then we invert Black's formula for this strike, i.e., solve

$$\mathbf{Cplt}(K) = \delta_k P(t, T)\mathbf{Black}(K, L_k(t), \sigma_k(K)\sqrt{T_{k-1} - t})$$

in $\sigma_k(K)$. Finally we change $K$ and repeat the process to get a curve $K \to \sigma_k(K)$.

There have been many works dealing with the volatility smiles. One popular approach is to start from an alternative dynamics, by assuming that under $T_k-$forward measure,

$$dL_k(t) = \xi_k(t, L_k(t))L_k(t)dW^{T_k}(t), \tag{34}$$

where $\xi_k$ can be either a deterministic or a stochastic function of $L_k(t)$. A deterministic $\xi_k$ leads to the so called 'local-volatility model'. For instance, we can take $\xi_k(t, L_k) = \xi_k(t)L_k(t)^{\beta-1}$, where $0 \leq \beta \leq 1$ and $\xi_k(t)$ is a deterministic function of $t$. The latter case leads to so called 'stochastic volatility models'. For example, we can take $\xi_k(t, L) = \xi_k(t)$, where $\xi_k(t)$ follows a stochastic differential equation. In this article, we will use the lognormal-mixture model introduced by Brigo et al.[9] as an example of local volatility models.

Several authors have attempted to modify the LMM to capture the implied volatility smile/skew. Stochastic processes more general than lognormal process have been proposed. For example, the constant elasticity of variance model(CEV) by Andersen-Andreasen[2] and the displaced diffusion model(DD) by Joshi-Rebonato[24] generate a monotone skew but not smile of implied volatility. A remedy to the above models has been to extend standard LMM by adopting mean reverting square root process for variance, such as Andersen and Brotherton-Ractliffe[4], Wu and Zhang [45], which produce additional curvature to the volatility curve. One of the main drawbacks of above models is that the volatility dynamics of all forward rates are driven by the same stochastic process, which may have difficulty in capturing different individual smile or skew shape of caps and swaptions. Hagan et al.[14] apply the SABR model to LIBOR modeling. However the volatility in the SABR model does not mean-revert.

## 3.2 Random Field Lognormal-mixture Volatility Model

In this section we present a local volatility model to capture the implied volatility smile. We extend the lognormal mixture model derived in Brigo et al. [9] to the random field case. Similar to Eq.(34), we can assume that under the $T_k$-forward measure, the dynamics of $L_k(t)$ in the random field setting is

$$dL_k(t) = \int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) dW^{T_k}(t,u)du, \tag{35}$$

where $\xi_k$ is a deterministic function of $L_k(t)$. Similar to the random field LIBOR market model, we have the following proposition.

**Proposition 3.1. (Random field local volatility dynamics under forward measures)** *Under the assumption of Eq.(35), the dynamics of $L_k(t)$ under the $T_j$-forward measure, in three cases $j < k, j = k, j > k$, are described respectively by the following equations*

$$\begin{cases} \dfrac{dL_k(t)}{L_k(t)} = \displaystyle\int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u)[dW^{T_j}(t,u) + \Lambda_j^k(t,u)dt]du, & j < k; \\ \dfrac{dL_k(t)}{L_k(t)} = \displaystyle\int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) dW^{T_j}(t,u)du, & j = k; \\ \dfrac{dL_k(t)}{L_k(t)} = \displaystyle\int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u)[dW^{T_j}(t,u) + \Lambda_k^j(t,u)dt]du, & j > k. \end{cases} \tag{36}$$

$$\Lambda_j^k(t,u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t) \xi_i(t, L_k(t), v) c(u,v)}{\delta_i L_i(t) + 1} dv,$$

*where $W^{T_j}(t,u)$ is a random field under the $T_j$-forward measure. The above equations admit a unique solution if the coefficients $\xi_k(\cdot,\cdot,\cdot)$ are locally bounded, locally Lipschitz continuous and predictable.*

Brigo and Mercurio [8] provide a class of analytical models based on a given mixture of densities. In this section we will extend them to the random field setting and provide closed-form formulas for caplet pricing and corresponding impled volatility that can be used for calibration.

Let us consider a series of diffusion process $G_i^k(t)$ with dynamics

$$dG_i^k(t) = G_i^k(t) \int_{T_{k-1}}^{T_k} \nu_i^k(t, G_i^k(t), u) dW^{T_k}(t, u) du, \tag{37}$$

with initial value $G_i^k(0) = L_k(0)$ for all $i = 1, 2, ..., M$, where $W^{T_k}(t, u)$ is a random field under the $T_k$-forward measure with correlation $c(t, T_1, T_2)$ as described in Eq.(7) and $\nu_i^k$ are real functions satisfying regular conditions to ensure existence and uniqueness o f the solutions to SDE 37. For each $t$, we denote the density function of $G_i^k$ by $p_i^k(x, t)$. Similarly to Brigo and Mercurio [8], the problem here is to derive the local volatility $\xi_k(t, u)$, for each time $t$, such that the density function $p^k(x, t)$ of $L_k(t)$ under $T_k$-forward measure satisfies

$$p^k(x, t) = \frac{d}{dx} P^{T_k}\{L_k(t) \leq x\} = \sum_{i=1}^{M} \omega_i \frac{d}{dx} P^{T_k}\{G_i^k(t) \leq x\} du = \sum_{i=1}^{M} \omega_i p_i^k(x, t),$$

where $\omega_i$ is a weighting function with $\sum_{i=1}^{M} \omega_i = 1$. In fact $p^k(x, t)$ is a proper density function under the $T_k$-forward measure since

$$\int_0^{+\infty} x p^k(x, t) dx = \int_0^{+\infty} x \sum_{i=1}^{M} \omega_i p_i^k(x, t) dx = \sum_{i=1}^{M} \omega_i G_i^k(0) = L_k(0),$$

if the conditions for exchange of integrals are verified. The last calculation follows from the fact that $G_i^k(t)$ is a martingale under the $T_k$-forward measure. We know that the local volatility $\xi_k(t, L_k(t))$ is

$$\int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} \xi_k(t, u) \xi_k(t, v) c(u, v) dv du = \frac{\sum_{i=1}^{M} \omega_i (\int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} \nu_i^k(t, u) \nu_i^k(t, v) c(u, v) dv du) p_i^k(x, t)}{\sum_{i=1}^{N} \omega_i p_i^k(x, t)}. \tag{38}$$

If we take $c(x, y) = 1$, the above formula reduces to

$$\xi_k(t) = \sqrt{\frac{\sum_{i=1}^{M} \omega_i \nu_i^k(t)^2 p_i^k(x, t)}{\sum_{i=1}^{N} \omega_i p_i^k(x, t)}}, \tag{39}$$

which is the original lognormal-mixture model derived in Brigo et al.[9].
Since

$$\int_{T_{k-1}}^{T_k} \xi_k(t, L_k(t), u) dW^{T_k}(t, u) du$$

is normally distributed with variance

$$\int_{T_{k-1}}^{T_k}\int_{T_{k-1}}^{T_k}\xi_k(t,L_k(t),u)\xi_k(t,L_k(t),x)c(u,v)dvdu,$$

we have the following proposition.

**Proposition 3.2. (Random field lognormal-mixture dynamics under forward measures)** *The dynamics of $L_k(t)$ is given by*

$$dL_k(t)=L_k(t)\sqrt{\frac{\sum_{i=1}^{M}\omega_i(\int_{T_{k-1}}^{T_k}\int_{T_{k-1}}^{T_k}\nu_i^k(t,u)\nu_i^k(t,v)c(u,v)dvdu)p_i^k(x,t)}{\sum_{i=1}^{M}\omega_i p_i^k(x,t)}}dW^{T_k}(t),\tag{40}$$

*where $W^{T_k}(t)$ is a Brownian motion under the $T_k$-forward measure.*

If we assume that in Eq.(37)

$$\nu_i^k(t,x,u)=\nu_i^k(t,u),\tag{41}$$

i.e., the densities $p_i^k(x,t)$ are all log-normal, where for all $k$, $\nu_i^k(t)$ are deterministic and continuous functions of time that are bounded from above and below by strictly positive constants, the marginal density of $G_i^k(t)$ is then log-normal:

$$p_i^k(x,t)=\frac{1}{x\nu_i^k(t)\sqrt{2\pi}}exp\{-\frac{1}{2\nu_i^k(t)^2}[\ln\frac{x}{L_k(t)}+\frac{1}{2}\nu_i^k(t)^2]^2\},\tag{42}$$

$$\nu_i^k(t) = \sqrt{\int_0^t\int_{T_{k-1}}^{T_k}\int_{T_{k-1}}^{T_k}\nu_i^k(s,u)\nu_i^k(s,v)c(s,u,v)dudvds}.\tag{43}$$

### 3.2.1 Option pricing in a random field local volatility model

In this section, we derive the closed-form pricing formulas for caplets in the random field lognormal-mixture model.

The payoff of a caplet at time $T_k$ is

$$\delta_k[L_k(T_{k-1})-K]^+.$$

The time $t$ price of the caplet is

$$\begin{aligned}\mathbf{Cplt}(t,K,T_{k-1},T_k) &= \delta_k P(t,T_k)E^{T_k}[[L_k(T_{k-1})-K]^+|\mathcal{F}_t]\\ &= P(0,T_k)\int_0^{+\infty}[x-K]^+p_k(x,t)dx\\ &= P(0,T_k)\sum_{i=1}^{M}\int_0^{+\infty}[x-K]^+p_i^k(x,t)dx.\end{aligned}\tag{44}$$

Suppose that $L_k(t)$ follows the dynamics of Eq.(35) under the $T_k$-forward measure and Eq.(41) holds, then the caplet price is

$$\mathbf{Cplt}(t, K, T_{k-1}, T_k) = \delta_k P(t, T_k) \sum_{i=1}^{M} \mathbf{Black}(K, L_k(t), \nu_i^k(T_{k-1})).$$

Given the above closed-form solution, we can derive an explicit approximation for the caplet implied volatility as a function of the strike price.

**Proposition 3.3. (Implied volatility of the random field lognormal-mixture model)** *Define* $m = \ln \frac{L_k(t)}{K}$. *The implied volatility* $\sigma_k^{Black,RF}$ *is*

$$\begin{aligned} & \sigma_k^{Black,RF}(m) \\ = \; & \sigma_k^{Black,RF}(0) + \frac{1}{2\sigma_k^{Black,RF}(0)(T_{k-1}-t)} \sum_{i=1}^{M} \omega_i [\frac{\sigma_k^{Black,RF}(0)\sqrt{T_{k-1}-t}}{\nu_i^k(T_{k-1})} \\ & e^{\frac{1}{8}(\sigma_k^{Black,RF}(0))^2(T_{k-1}-t)-\nu_i^k(T_{k-1})^2} - 1]m^2 + o(m^2), \end{aligned} \tag{45}$$

*where the at-the-money caplet implied volatility* $\sigma_k^{Black,RF}(0)$ *is*

$$\sigma_k^{Black,RF}(0) = \frac{2}{\sqrt{T_{k-1}-t}} \Phi^{-1}(\sum_{i=1}^{M} \omega_i \Phi(\frac{1}{2}\nu_i^k(T_{k-1}))). \tag{46}$$

Proposition 3.3 can be used to capture the volatility smile when calibrating the random field lognormal-mixture model to caplets.

# 4 Calibration

The models we discuss and derive in this paper require in general three different inputs, the initial forward curve, the instantaneous volatilities of the forward rates, and the correlation structure. The instantaneous volatilities of forward rates are usually assumed to depend on current time $t$, forward rate maturity $T$ and time to maturity $T-t$. The most popular assumption is that the volatilities depend only on time to maturity (time-homogenous). There are two ways to specify the volatility function. One is to assume that volatility is piecewise-constant. Another way is to assume some parametric form for the volatility. In this paper we assume that the instantaneous volatility is of a particular parametric form with several time independent parameters. The estimation of the correlation structure can be based on historically estimated correlation matrix or some parametric form. However, the historical matrix is very volatile and is likely to have some outliers. Thus we follow the other approach to assume a parametric functional form for the correlation structure. See Sec.4.1 for details of the specification of model inputs.

### 4.1 Model specification

We need to calibrate the instantaneous volatility functions $\xi_k(t)$, $k = 1, 2, ..., N$ and the correlation matrix $\rho_{i,j}(t)$, $i, j = 1, 2, .., N$ from the data of the initial yield curve $L_k(0)$ and the caps and swaptions prices observed in the market. Notice that in the random field case, the correlation structure may take the continuum form $c(u, v)$. In this paper, we choose to parameterize the instantaneous volatility $\xi_k(t)$ and the correlation structure $c(u, v)$.

From , the volatility $\xi_k(t)$ can be simply taken as $\xi_k(t) = f(t, T_k)h(t)g(T_k)$ for the instantaneous volatility, while $h(t), g(T_k)$ are usually taken to be 1 and

$$f(t, T_k) = ae^{-b(T_k - t)}; a, b > 0. \tag{47}$$

The correlation structure differs for LMM and random field LMM. For the LMM, one need to specify the functional form of correlation matrix $\rho_{i,j}$, while for the random field LMM, one need to specify the correlation function $c(u, v)$. The difference of correlation structures for the LMM and the random field LMM is discussed as follows.

For the LMM, given the dynamics in Eq.(25) we know that the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is defined as

$$\frac{Cov(dL_i(t), dL_j(t))}{\sqrt{Var(dL_i(t))Var(dL_j(t))}} = \frac{\xi_i(t)\xi_j(t)dW_i(t)dW_j(t)}{\sqrt{\xi_i^2(t)\xi_j^2(t)}} = dW_i(t)dW_j(t) = \rho_{i,j}(t),$$

which means that the instantaneous correlation of forward rates is exactly the same as the correlation structure of the Brownian motion $W(t)$. However, for the random field LMM, given the dynamics in Eq.(24), the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is defined as

$$\begin{aligned} &\frac{Cov(dL_i(t), dL_j(t))}{\sqrt{Var(dL_i(t))Var(dL_j(t))}} \\ = &\frac{\int_{T_{i-1}}^{T_i}\int_{T_{j-1}}^{T_j} \xi_i(t,x)\xi_j(t,y)c(x,y)dxdy}{\sqrt{\int_{T_{i-1}}^{T_i}\int_{T_{i-1}}^{T_i} \xi_i(t,x)\xi_i(t,y)c(x,y)dxdy \int_{T_{j-1}}^{T_j}\int_{T_{j-1}}^{T_j} \xi_j(t,x)\xi_j(t,y)c(x,y)dxdy}}, \end{aligned}$$

which indicates that the instantaneous correlation of forward rates depends on both the correlation structure and the instantaneous volatilities. In other words, the correlation matrix depends the specification of both $\xi(t, y)$ and $c(x, y)$. The number of factors in the LIBOR market model depends on the rank of correlation matrix $\rho$. A rank-$d$ correlation matrix $\rho$ with entries defined in Eq.(49) gives rise to a $d$-factor LIBOR market model. In this paper we take a full rank correlation matrix and thus the LIBOR market model considered in this paper has the number of factors the same as the number of forward rates considered. Notice that if we specify the correlation directly without parametrization, we need to calibrate $K(K + 1)/2$

parameters, where $K$ is the size of the correlation matrix. The number of parameters becomes very large if $K$ is large. Thus the techniques of factor reduction will be needed to reduce the rank of the matrix and thus the number of parameters. On the other hand, we parameterize the correlation structure. The parameters needed to be calibrated are just the parameters in the functional form. Thus it is not necessary to reduce the correlation matrix to lower rank.

We take the following correlation structure

$$c(T_i, T_j) := e^{-\rho_\infty |T_i - T_j|}. \tag{48}$$

For the Brownian motion case, we take the matrix form

$$\rho_{i,j} = e^{-\rho_\infty \frac{|i-j|}{N}}, \tag{49}$$

where $\frac{1}{N} = T_i - T_{i-1}$, for $i = 1, 2, ...$ The correlation is independent of current time $t$.

## 4.2 Calibration Procedure

In this section, we provide the closed-form Black implied volatility for caplets and swaptions, as functions of the instantaneous volatility $\xi(t, T)$ and the correlation structure $c(u, v)$. The Black implied volatilities for caplets are derived in Eq.(45), where $\nu_i^k(t)$ takes different form for the LMM and the RFLMM. For the LMM,

$$\nu_i^k(t) = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} f_i^2(s, T_k) ds}. \tag{50}$$

For the RFLMM,

$$\nu_i^{k,RF}(t) = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} [\int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} f_i(s, x) f_i(s, y) c(x, y) dx dy] ds},$$

where $f_i(s, T)$ takes the form of Eq.(47). To facilitate calibration and comparison between the LMM and the RFLMM, it is reasonable to set $f(t, x) = f(t, T_k)$ for $x \in [T_{k-1}, T_k]$. Thus the above equation becomes

$$\begin{aligned} \nu_i^{k,RF}(t) &= \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \delta_k^2 f_i^2(s, T_k) ds \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c(x, y) dx dy} \\ &= \sqrt{\frac{1}{T_{k-1} - t} c_{kk} \int_t^{T_{k-1}} \delta_k^2 f_i^2(s, T_k) ds}, \end{aligned} \tag{51}$$

where $c_{kk} = \int_{T_{k-1}}^{T_k} \int_{T_{k-1}}^{T_k} c(x, y) dx dy$.

The Black implied volatilities for swaptions are derived as follows. From Eq.(33) and Eq.(48), we know that correlation structure is independent of current time $t$, thus we have

$$
\begin{aligned}
&\overline{\sigma}_{i,j}^{Black,RF} \\
=&\sqrt{\frac{1}{T_i-t}\int_t^{T_i}\|\sum_{k=i+1}^{j}\frac{\delta_k L_k(s)\gamma_k^{i,j}(s)}{1+\delta_k L_k(s)}\int_{T_{k-1}}^{T_k} f(s,T_k)dW(t,u)\|^2 ds} \\
=&\sqrt{\frac{1}{T_i-t}\sum_{k,l=i+1}^{j}\frac{\delta_k L_k(t)\gamma_k^{i,j}(t)}{1+\delta_k L_k(t)}\frac{\delta_l L_l(t)\gamma_l^{i,j}(t)}{1+\delta_l L_l(t)}\int_t^{T_i} c_{kl}\delta_k\delta_l f(s,T_k)f(s,T_l)ds} \\
=&\sqrt{\frac{1}{T_i-t}\sum_{k,l=i+1}^{j}\frac{\delta_k L_k(t)\gamma_k^{i,j}(t)}{1+\delta_k L_k(t)}\frac{\delta_l L_l(t)\gamma_l^{i,j}(t)}{1+\delta_l L_l(t)}\sigma_k^{Black,RF}\sigma_l^{Black,RF}\frac{c_{kl}}{\sqrt{c_{kk}c_{ll}}}\Theta_{l,k}(t)},
\end{aligned}
\tag{52}
$$

where

$$
\Theta_{l,k}(t) = \frac{\sqrt{T_l-t}\sqrt{T_k-t}\int_t^{T_i} f(s,T_k)f(s,T_l)ds}{\sqrt{\int_t^{T_k} f^2(s,T_k)ds}\sqrt{\int_t^{T_l} f^2(s,T_l)ds}}. \tag{53}
$$

The second equation above is obtained by using standard freezing approximation techniques, i.e. approximately evaluating the LIBOR rates $L_k(s)$, $t \le s \le T_i$, appearing in the instantaneous volatility at initial time $t$.

We conduct two calibrations. The first calibration is for the lognormal mixture models, using caplet volatility surface, by minimizing the root of mean square percentage error

$$
RMSPE_{cplt} = \sqrt{\frac{1}{NN_s}\sum_{i=1}^{N_s}\sum_{j=1}^{N}\left(\frac{\sigma_j^{Black}(m_i)-\sigma_j^{Market}(m_i)}{\sigma_j^{Market}(m_i)}\right)^2}, \tag{54}
$$

where $N$ is the number of tenor days and $N_s$ is the number of caplet strike price used in calibration.

The second calibration is for lognormal models, using both at-the-money swaption and cap volatilities, by minimizing the root of mean square percentage error

$$
RMSPE_{swpt} = \sqrt{\frac{2}{(2N-M-1)M}\sum_{i=1}^{M}\sum_{j=i+1}^{N}\left(\frac{\overline{\sigma}_{i,j}^{Black}-\overline{\sigma}_{i,j}^{Market}}{\overline{\sigma}_{i,j}^{Market}}\right)^2}. \tag{55}
$$

where $N$ is the number of tenor days and $M$ is the number of swaption maturities used in calibration. The second calibration is considered joint-calibration to both swaption and caps because the market quotes of Black's volatility for caps enter into swaption volatility through Eq. (52). Calibration results are shown in Sec.4.3.

### 4.3 Numerical Results

In this section we present the calibration results of LMM and RFLMM. The input of the calibration consists of market closing prices on September 29, 2006 from Bloomberg. The data include annualized initial forward rate curve, annualized caplet volatilities and swaption volatilities.

#### 4.3.1 Parameter calibration

Firstly, we investigate the caps surface calibration using the lognormal-mixture model. The caps considered here have floating leg 6M LIBOR, maturities from 1 to 30 years and strikes from 1% to 10%. Table 1 and 2 show the calibration results on September 29, 2006. The column "RMSPE" represents the root mean squared percentage error of the model volatilities and market quoted volatilities.

Table 1: Calibration Results for LMM(Cap Surface)

| $\omega$ | a | b | RMSPE(%) |
|---|---|---|---|
| $\omega_1$=0.23 | 1.61 | 1.14 | - |
| $\omega_2$=0.25 | 1.03 | 1.18 | - |
| $\omega_3$=0.24 | 0.103 | 0.38 | - |
| $\omega_4$=0.28 | 0.004 | 0.029 | 7.29 |

Table 2: Calibration Results for RFLMM(Cap Surface)

| $\omega$ | a | b | $\rho_\infty$ | RMSPE(%) |
|---|---|---|---|---|
| $\omega_1$=0.16 | 0.728 | 1.21 | 0.705 | - |
| $\omega_2$=0.15 | 0.177 | 1.02 | 0.332 | - |
| $\omega_3$=0.36 | 0.023 | 0.037 | 0.073 | - |
| $\omega_4$=0.33 | 0.002 | 0.015 | 0.014 | 6.07 |

From Table 1 and 2, we notice that random field models produce smaller RMSPE, compared to the Brownian motion LMM. The reason is perhaps that the calibration of cap volatility surface in LMM does not utilize the information from the correlation structure, while the random field model uses correlation structure as one critical input. This makes the cap calibration more accurate in the random field model.

We plot the calibrated volatility smile for maturity 1 year, 3 year and 15 year in Figure 1. From these figures we observe that the random field lognormal-mixture LMM fits the smile/skew better than the original lognormal-mixture LMM.

Secondly, we consider the joint calibration of the lognormal models to at-the-money swaptions and caps. The swaptions considered here have combined option maturity and underlying swap maturity not exceeding 10 years. Table 3 shows the calibration results on Sept. 29, 2006.

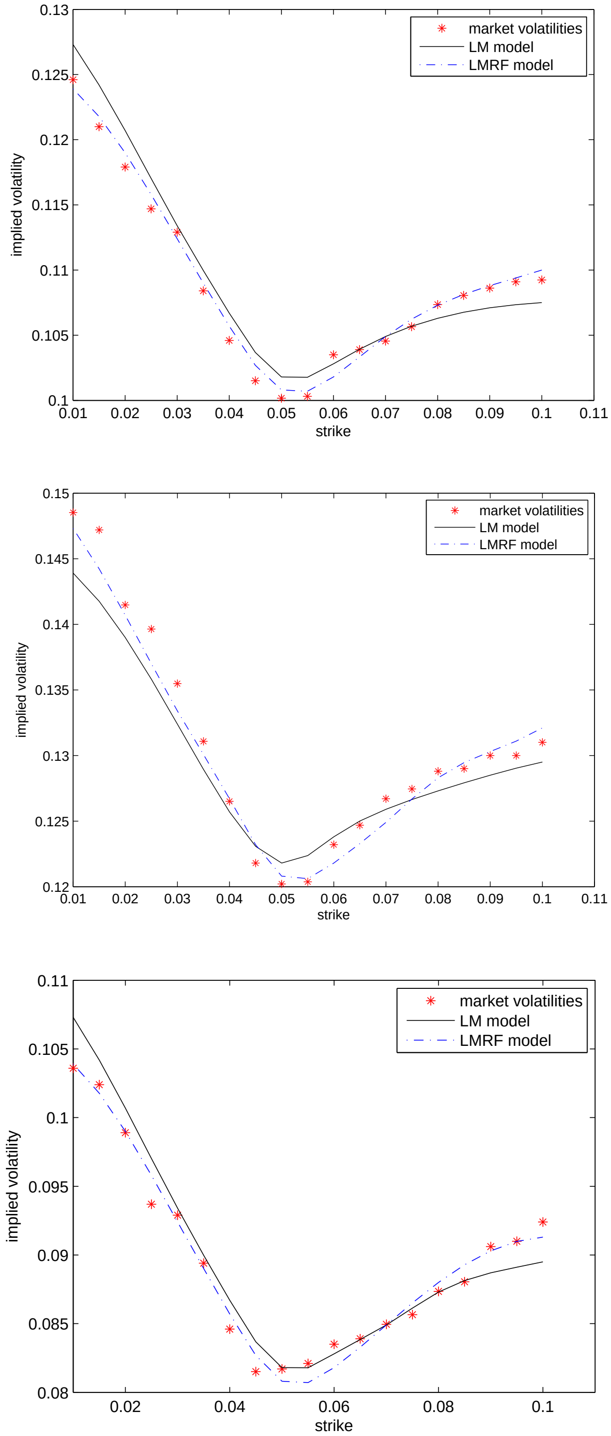


Figure 1: Examples of Caplet Volatility Smile Calibration for Maturity 1 year, 3 years, 15 years.

Table 3: Calibration Results for LMM and RFLMM(Swaption)

| | a | b | $\rho_\infty$ | RMSPE(%) |
|---|---|---|---|---|
| LMM | 1.82 | 1.43 | 1.56 | 5.93 |
| RFLMM | 0.88 | 1.13 | 1.79 | 5.01 |

From Table 3, we observe that the random field model can fit jointly the market prices of ATM caps and swaptions better than the Brownian model.

# 5 Conclusions

This paper extends the LIBOR market model by modeling forward rate innovations via a random field. Derivatives prices are shown to satisfy a Black-Scholes type partial differential equation, which gives rise to closed-form solutions for caplet and swaption prices. We then derive a local volatility smile model in the random field setting. We use the lognormal-mixture model as an example. Approximation formulas for option implied volatilities are obtained. Finally we discuss the calibration procedure of the random field LIBOR market model. Our calibration results indicate that the random field LIBOR market model is potentially superior in capturing market prices of caps and swaptions, in addition to possessing other documented advantages of random field models (no need of frequent recalibration or to specify the number of factors in advance).

# 6 Appendix

**Appendix 1. Proof of Proposition 2.1***:*

*Proof.* Given the dynamics of the zero coupon bond price $P(t,T)$ in Eq.(15), by It$\hat{o}$'s formula, we have that

$$d(\frac{1}{P(t,T_k)}) = -\frac{r(t)dt - \int_t^{T_k}\sigma(t,u)d\widetilde{W}(t,u)du + (\int_t^{T_k}\sigma(t,u)d\widetilde{W}(t,u)du)^2}{P(t,T_k)},$$

and

$$dP(t,T_{k-1})d(\frac{1}{P(t,T_k)}) = \frac{-P(t,T_{k-1})}{P(t,T_k)}\int_t^{T_{k-1}}\sigma(t,u)d\widetilde{W}(t,u)du\int_t^{T_k}\sigma(t,u)d\widetilde{W}(t,u)du.$$

Thus the dynamics of $L_k(t)$ under risk neutral measure Q can be derived as

$$\begin{aligned}
dL_k(t) &= d[\frac{1}{\delta_k}(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1)] \\
&= \frac{1}{\delta_k}[\frac{dP(t,T_{k-1})}{P(t,T_k)} + P(t,T_{k-1})d(\frac{1}{P(t,T_k)}) + dP(t,T_{k-1})d(\frac{1}{P(t,T_k)})] \\
&= \frac{1}{\delta_k}\frac{P(t,T_{k-1})}{P(t,T_k)}[\int_{T_{k-1}}^{T_k}\sigma(t,u)d\widetilde{W}(t,u)du + \int_t^{T_k}\sigma(t,u)d\widetilde{W}(t,u)du \\
&\quad \int_{T_{k-1}}^{T_k}\sigma(t,u)d\widetilde{W}(t,u)du].
\end{aligned}$$

Now we derive the dynamics of forward rates $L_k(t)$ under $T_k$-forward measure. Suppose that there exists a function $\theta(t,T_k,u)$ such that $dW^{T_k}(t,u) := \theta(t,T_k,u)dt + d\widetilde{W}(t,u)$ has normal distribution $\Phi(0,dt)$ under $T_k$-forward measure. Replace $d\widetilde{W}(t,u)$ by $dW^{T_k}(t,u) - \theta(t,T_k,u)dt$ in above formula, we obtain

$$\begin{aligned}
dL_k(t) \quad &= \frac{1}{\delta_k}\frac{P(t,T_{k-1})}{P(t,T_k)}\{\int_{T_{k-1}}^{T_k}\sigma(t,u)[dW^{T_k}(t,u) - \theta(t,T_k,u)dt]du + \int_t^{T_k}\sigma(t,u) \\
&\quad [dW^{T_k}(t,u) - \theta(t,T_k,u)dt]du\int_{T_{k-1}}^{T_k}\sigma(t,u)[dW^{T_k}(t,u) - \theta(t,T_k,u)dt]du\}.
\end{aligned}$$

Since $L_k(t)$ is a martingale under $T_k$-forward measure, the drift term should vanish, i.e.

$$\begin{aligned}
\int_{T_{k-1}}^{T_k}\sigma(t,u)\theta(t,T_k,u)dtdu &= \int_{T_{k-1}}^{T_k}\sigma(t,u)dW^{T_k}(t,u)du\int_t^{T_k}\sigma(t,v)dW^{T_k}(t,v)dv \\
&= \int_{T_{k-1}}^{T_k}\sigma(t,u)[\int_t^{T_k}\sigma(t,v)c(u,v)dtdv]du,
\end{aligned}$$

which means that

$$\theta(t, T_k, u) = \int_t^{T_k} \sigma(t,v)c(u,v)dv.$$

This completes the proof. $\square$

**Appendix 2. Proof of Proposition 2.4:**

*Proof.* From Eq. (18) and Corollary. 2.2,

$$dW^{T_k}(t,u) - \int_t^{T_k} \sigma(t,v)c(u,v)dvdt = dW(t,u) = dW^{T_{k+1}}(t,u) - \int_t^{T_{k+1}} \sigma(t,v)c(u,v)dvdt.$$

Thus

$$dW^{T_k}(t,u) = dW^{T_{k+1}}(t,u) - \int_{T_k}^{T_{k+1}} \sigma(t,v)c(u,v)dvdt,$$

from which follows that for $j > k$

$$dW^{T_k}(t,u) = dW^{T_j}(t,u) - \sum_{i=k+1}^{j} \int_{T_{i-1}}^{T_i} \sigma(t,v)c(u,v)dvdt. \tag{56}$$

Plugging Eq. (56) in to Eq. (18), we obtain

$$\begin{aligned} dL_k(t) &= \frac{1}{\delta_k}(\delta_k L_k + 1)[\int_{T_{k-1}}^{T_k} \sigma(t,u)dW^{T_j}(t,u)du \\ &\quad - \sum_{i=k+1}^{j} \int_{T_{k-1}}^{T_k} \sigma(t,u) \int_{T_{i-1}}^{T_i} \delta_k \sigma(t,v)c(u,v)dvdudt]. \end{aligned}$$

$L_k(t)$ is a martingale under the $T_k$-forward measure. By the random field martingale representation proposition, there exists a function $\xi(t,u)$ such that

$$dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)dW^{T_k}(t,u)du. \tag{57}$$

From Eq.(18),

$$\xi_k(t,u) = \frac{\delta L_k(t) + 1}{\delta L_k(t)} \sigma(t,u). \tag{58}$$

Thus Eq.(57) becomes

$$\begin{aligned} dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)dW^{T_j}(t,u)du \\ &\quad - \sum_{i=k+1}^{j} L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u) \int_{T_{i-1}}^{T_i} \frac{\delta L_i(t)\xi_i(t,v)c(u,v)}{\delta L_i(t)+1} dvdudt] \\ &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)[dW^{T_j}(t,u) + \sum_{i=k+1}^{j} \int_{T_{i-1}}^{T_i} \frac{\delta L_i(t)\xi_i(t,v)c(u,v)}{\delta L_i(t)+1} dvdt]du. \end{aligned}$$

The derivation in case $j < k$ is similar. The existence and uniqueness of solution $L_k(t)$ is assured by the existence and uniqueness of $f(t,T)$ in Eq.(8). Thus, given the coefficients satisfying the required conditions, locally bounded, locally Lipschitz continuous and predictable, there exists a unique $f(t,T)$ for Eq.(8). This completes the proof. □

**Appendix 3. Proof of Proposition 2.5:**

*Proof.* If the value of $dW^T(t,u)$ on $[T_{k-1}, T_k]$ is $dW_k^T(t)$, then

$$\int_{T_{k-1}}^{T_k} \int_{T_{i-1}}^{T_i} c(u,v)dvdu = \delta_k \delta_i c(t, T_k, T_i),$$

which means that for $j < k$, Proposition 2.4 becomes

$$\begin{aligned} dL_k(t) &= L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)[dW^{T_j}(t,u) + \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t)\xi_i(t,v)c(u,v)}{\delta_i L_i(t)+1} dvdt]du \\ &= L_k(t)\delta_k \xi_k(t,u)[dW^{T_j}(t,u) + \sum_{i=j+1}^{k} \delta_i \frac{\delta_i L_i(t)\delta_i \xi_i(t,v)c(t,T_i,T_k)}{\delta_i L_i(t)+1} dt]. \end{aligned}$$

From Eq.(25), we can easily take $c(t, T_i, T_k) = \rho_{i,k}$ and take $\delta_k \xi_k(t)$ to be $\xi_k(t)$. The derivation for $j > k$ is analogous. Thus Eq.(24) in Proposition 2.4 will reduce to Eq.(6) and Proposition 2.4 reduces to Eq.(25), which means that Eq.(25) is a discrete case of Proposition 2.4. This completes the proof. □

**Appendix 4. Proof of Proposition 2.6 (Random field Black-Scholes equation)***:*

*Proof.* We provide the proof of the Black-Scholes equation with random field for time dependent parameters. Suppose that we have an option $V$ on some underlying asset $S$, which has dynamics

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \int_{t_1}^{t_2} \sigma(t,u)dW(t,u)du. \tag{59}$$

We create a portfolio $\Pi$ which consists of long $\Delta$ number of the asset and short one options, i.e.

$$\Pi = -V + \Delta S.$$

The increment of portfolio value is given by that of option and underlying asset, i.e.

$$d\Pi = -dV + \Delta dS.$$

By Itó's formula for $dV$ and the dynamics of asset $S$, we can have that

$$\begin{aligned} d\Pi &= -dV + \Delta dS \\ &= -\frac{\partial V}{\partial t}dt - \frac{1}{2}[\int_{t_1}^{t_2} \sigma(t,u)dW(t,u)du \int_{t_1}^{t_2} \sigma(t,u)dW(t,u)du]S^2\frac{\partial^2 V}{\partial S^2} + (\Delta - \frac{\partial V}{\partial S})dS \\ &= (-\frac{\partial V}{\partial t} - \frac{1}{2}[\int_{t_1}^{t_2}\int_{t_1}^{t_2} \sigma(t,u)\sigma(t,v)c(u,v)dudv]S^2\frac{\partial^2 V}{\partial S^2})dt + (\Delta - \frac{\partial V}{\partial S})dS \\ &= (-\frac{\partial V}{\partial t} - \frac{1}{2}\widehat{\sigma}^2(t,t_1,t_2)S^2\frac{\partial^2 V}{\partial S^2})dt + (\Delta - \frac{\partial V}{\partial S})dS, \end{aligned}$$

where

$$\widehat{\sigma}^2(t,t_1,t_2) = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \sigma(t,u)\sigma(t,v)c(u,v)dudv. \tag{60}$$

The portfolio will become non-stochastic if we choose $\Delta = \frac{\partial V}{\partial S}$, which means that the portfolio will grow in risk free interest rate $r(t)$:

$$d\Pi = (-\frac{\partial V}{\partial t} - \frac{1}{2}\widehat{\sigma}^2(t,t_1,t_2)S^2\frac{\partial^2 V}{\partial S^2})dt = r(t)(-V + S\frac{\partial V}{\partial S})dt.$$

By rearranging the equation above we can have the Black-Scholes equation with random field for time dependent parameters:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\widehat{\sigma}^2(t,t_1,t_2)S^2\frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0.$$

We can introduce new parameters $Y = \ln S$,$\tau = T - t$, $U = e^{\int_0^\tau r(t)dt}V$ and the above equation becomes

$$\frac{\partial U}{\partial \tau} = \frac{\int_{t_1}^{t_2}\int_{t_1}^{t_2} \sigma(t,u)\sigma(t,v)c(u,v)dudv}{2}\frac{\partial^2 U}{\partial Y^2} + [r(\tau) - \frac{\int_{t_1}^{t_2}\int_{t_1}^{t_2} \sigma(t,u)\sigma(t,v)c(u,v)dudv}{2}]\frac{\partial U}{\partial Y},$$

which has fundamental solution as

$$\Phi(Y,\tau) = \frac{1}{\sqrt{2\pi\int_0^\tau \widehat{\sigma}^2(u,t_1,t_2)du}}e^{-[Y + \int_0^\tau (r(u) - \widehat{\sigma}^2(u,t_1,t_2)/2)du]^2/[2\int_0^\tau \widehat{\sigma}^2(u,t_1,t_2)du]}.$$

Thus the solution of Black-Scholes equation is given by

$$U(Y,\tau) = \int_{-\infty}^{+\infty} U(v,0)\Phi(Y - v,\tau)dv.$$

The terminal conditions $V(S,T) = \max(S_T - K, 0)$ for call option give the price for a call option as

$$C = SN(\tilde{d}_1) - Ke^{-\int_0^\tau r(u)duN(\tilde{d}_2)},$$

where

$$\tilde{d}_1 = \frac{1}{\sqrt{\int_0^\tau \widehat{\sigma}^2(u,t_1,t_2)du}}[\ln\frac{S}{K} + \int_0^\tau (r(u) + \frac{\widehat{\sigma}^2(u,t_1,t_2)}{2})du], \quad (61)$$

$$\tilde{d}_2 = \tilde{d}_1 - \sqrt{\int_0^\tau \widehat{\sigma}^2(u,t_1,t_2)du}. \quad (62)$$

This completes the proof. □

## References


[1] R. J. Adler, Geometry of random fields, *Wiley*, 1981.

[2] L. Andersen and J. Andreasen, Volatility skews and extension of the LIBOR market model, *Applied Mathematical Finance*, 7 (1), 1-32, 2000.

[3] L. Andersen and J. Andreasen, Volatile volatilities, *Risk Magazine*, 15 (12), 163-168, 2000.

[4] L. Andersen and R. Brotherton-Ratchilffe, Extended LIBOR market model with stochastic volatility, *Journal of Computational Finance*, 9 (1), 1-40, 2005.

[5] C. A. Bester, Random field and affine models for interest rates: an empirical comparison, *Working Paper*, 2004.

[6] F. Black, The Pricing of commedity contracts, *Journal of Financial Economics*, 3, 167-179, 1976.

[7] A. Brace and D. Gatarek and M. Musiela, The market model of interest rate dynamics, *Mathematical Finance*, 7, 127-155, 1997.

[8] D. Brigo and F. Mercurio, Interest rate models: theory and practice, with smile, inflation and credit, *Springer Finance*, New York, 2006.

[9] D. Brigo and F. Mercurio and M. Morini, Lognormal-mixture dynamics and calibration to market volatility smiles, *Internatial Journal of Theoretical and Applied Finance*, 5 (4), 427-446, 2002.

[10] A. Buraschi and F. Corielli, Risk management implications of time-inconsistency: model updating and recalibration of no-arbitrage models, *Journal of Banking and Finance*, 29, 2883-2907, 2005.

[11] J. C. Cox and J. E. Ingesoll, and S. A. Ross, A Theory of the term strucure of interest rates, *Econometrica*, 53, 385-407, 1985.

[12] I. I. Gikhman and A. V. Skorokhod, Introduction to the theory of random processes, *Dover Publications*, 1996.

[13] R. S. Goldstein, The term structure of interest rates as a random field, *Review of Financial Studies*, 13, 365-284, 2000.

[14] P. Hagan and D. Kumar and A. Lesniewski and D. Woodward, Managing smile risk, *Wilmott Magazine*, 9, 84-108, 2002.

[15] P. Hagan and A. Lesniewski, LIBOR market model with SABR style stochastic volatility, *Working Paper*, 2008.

[16] D. Heath and R. A. Jarrow and A. J. Morton, Bond pricing and the term structure of interest rates, *Econometrica*, 60 (1), 77-105, 1992.

[17] P. Henry-labordere, Solvable local and stochastic volatility models: supersymmetric methods in option pricing, *Quantitative Finance*, 7 (5), 525-535, 2007.

[18] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies*, 6, 327-343, 1993.

[19] T. Y. Ho and S. B. Lee, Term structure movements and the pricing of interest rate contigent claims, *Journal of Finance*, 41, 1011-1029, 1986.

[20] J. Hull and A. White, The pricing of options on assets with stochastic volatilities, *Journal of Financial and Quantitative Analysis*, 3, 281-300, 1987.

[21] J. Hull and A. White, Pricing interest rate derivative securities, *Review of Financial Studies*, 3, 573-592, 1990.

[22] J. Hull and A. White, Forward rate volatilities, swap rate volatilities, and the implementation of the LIBOR market model, *Journal of Fixed Income*, 10, 46-62, 2000.

[23] F. Jamshidian, LIBOR and swap market models and measures, *Finance and Stochastics*, 1, 293-330, 1997.

[24] M. S. Joshi and R. Rebonato, A stochastic volatility displaced-diffusion extension of the LIBOR market model, *Quantitative Finance*, 6, 458-469, 2003.

[25] D. P. Kennedy, The term structure of interest rates as a Gaussian random field, *Mathematical Finance*, 4, 247-258, 1994.

[26] D. P. Kennedy, Characterizing Gaussian models of the term structure of interest rates, *Mathematical Finance*, 7, 107-118, 1997.

[27] J. Kerkhof and A. Pelsser, Observational equivalence of discrete string models and market models, *Journal of Derivatives*, 10 (1), 55-61, 2000.

[28] D. Khoshnevisan, Multiparameter process: an introduction to random fields, *Springer-Verlag*, New York, 2002.

[29] H. Li and F. Zhao, Unspanned stochastic volatility: evidence from hedging interest rate derivatives, *Journal of Finance*, 61, 314-378, 2006.

[30] F. Longstaff, P. Santa-Clara, and E. Schwartz, The relative valuation of caps and swaptions: theory and evidence, *Journal of Finance*, 56, 2067-2109, 2001.

[31] F. Mercurio, Interest rates and the credit crunch: new formulas and market models, *Bloomberg Portfolio Research*, FRONTIERS, 2009.

[32] R. K. Miltersen and K. Sandmann and D. Sondermann, Closed form solution for term sturcture derivatives with lognormal interest rates, *Journal of Finance*, 52, 409-430, 1997.

[33] M. Morini and N. Webber, An EZI method to reduce the rank of a correaltion matrix, *Applied Mathematical Finance*, 13 (4), 309-331, 2006.

[34] B. Oksendal, Stochastic differential equations: an introduction with applications, *Springer*, 2005.

[35] K. Pang and C. Wharf, Calibration of Gaussian Heath, Jarrow and Morton and random field interest rate term structure models, *Review of Derivatives Research*, 2 (4), 315-345, 1998.

[36] N. D. Pearson and T. S. Sun, Exploiting the conditional density in estimating the term structure: an application to the cox, ingersoll, and ross model, *Journal of Finance*, 49, 1279-1304, 1994.

[37] V. V. Piterbarg, Stochastic volatility model with time-dependent skew, *Applied Mathematical Finance*, 12 (2), 147-185, 2005.

[38] R. Rebonato, Calibrating the BGM model, *Risk Magazine*, 12, 74-79, 1999.

[39] R. Rebonato, Modern pricing of interest rate derivatives: the LIBOR market model and beyond, *Princeton University Press*, 2002.

[40] R. Rebonato, A time-homogeneous, SABR-consistent extension of the LMM: calibration and numerical results, *Risk Magazine*, 102-106, November 2007.

[41] P. Santa-Clara and D. Sornette, The dynamics of the forward interest rate curve with stochastic string model shocks, *Review of Financial Studies*, 14, 149-185, 2001.

[42] J. Schoenmakers and C. Coffey, Stable implied calibration of a multi-factor LIBOR model via a semi-parametic correlation structure, *Weierstrass Institute*, 2000.

[43] A. B. Trolle and E. S. Schwartz, A general stochastic volatility model for the pricing of interest rate derivatives, *Review of Financial Studies*, 22, 2007-2057, 2009.

[44] P. Vasicek, An equilibrim characterization of the term structure, *Journal of Financial Economics*, 5, 177-188, 1977.

[45] L. Wu and F. Zhang, LIBOR market model with stochastic volatility, *J. Industrial and Management Optimization*, 2 (2), 199-227, 2006.

[46] T. Wu, Pricing and hedging the smile with SABR: evidence from the interest rate caps market, *Journal of Futures Markets*, 32, 773-791, 2012.