Institut canadien des dérivés
Canadian Derivatives Institute

L'Institut bénéficie du soutien financier de l'Autorité des marchés financiers ainsi que du ministère des Finances du Québec

Document de recherche

DR 17-04

Longevity Bond Pricing in Equilibrium

Mai 2017

Ce document de recherche a été rédigée par :

Petar Jevtié, McMaster University
Minsuk Kwak, Hankuk University of Foreign Studies, Korea
Traian A. Pirvu, McMaster University

Longevity bond pricing in equilibrium[1]

Petar Jevtić
Department of Mathematics & Statistics
McMaster University
1280 Main Street West
Hamilton, ON, L8S 4K1
pjevtic@math.mcmaster.ca

Minsuk Kwak
Department of Mathematics
Hankuk University of Foreign Studies
81, Oedae-ro, Mohyeon-myeon, Cheoin-gu,
Yongin-si, Gyeonggi-do 449-791, Korea
mkwak@hufs.ac.kr

Traian A. Pirvu
Department of Mathematics & Statistics
McMaster University
1280 Main Street West
Hamilton, ON, L8S 4K1
tpirvu@math.mcmaster.ca

Abstract

We consider a partial equilibrium model for pricing a longevity linked bond in a model with stochastic mortality intensity that affects the income of economic agents. The agents trade in a risky financial security and in the longevity linked bond in order to maximize their utilities. Agent's risk preferences are of monetary type and are described by BSDEs (backward stochastic differential equations). The endogeneous equilibrium bond price is characterized by a BSDE. By using Clark-Haussmann formula, we prove that our longevity bond completes the market.

[1]This work is supported through IFSID grant "Securitization, Pricing, and Management of Longevity Risks and Derivatives". Minsuk Kwak acknowledges that this work was supported by Hankuk University of Foreign Studies Research Fund of 2016.

1 Introduction

When contrasted with the rest of human history, the 20th and 21st centuries have been witness to a significant increase in human life expectancy. This phenomenon is having an increasingly visible impact on private as well as public institutions of developed economies concerned with pension arrangements. In this context, there exists a risk that liabilities related to previous financial commitments grow higher than expected due to the unreliability of forecasts of mortality trends. This risk is called *longevity risk* ; essentially, it is the risk that lives in a reference population might, on average, live longer than anticipated. It is clear that providers of pension plans, annuity, and social insurance share a strong interest in proper assessment and management of this risk. Whereas assessing longevity risk is intimately connected with the development of better mortality models with stronger forecasting capabilities, its management, in order to facilitate transfer of longevity risk to capital markets, is at least in part dependent on the development of mortality linked financial instruments, such as mortality bonds, swaps, and mortality options type derivatives.

Although the market for mortality linked instruments, the so-called *life market*, is still far from being fully developed and is constantly experiencing ups and downs with respect to its growth, it is nonetheless demonstrating steady progress in its depth and scope of transactions. A highly useful and detailed account of this evolution can be found in Tan et al. (2015). As insightfully articulated in Bauer et al. (2010), the development of the "life market" implicitly entails questions about efficient engineering of mortality-linked securities or derivatives as well as their pricing both from conceptual and practical aspects.

In this paper, we focus on the problem of the pricing of longevity bonds. Longevity bonds can be viewed as basic mortality linked instruments, where payments are made to be contingent on the proportion of cohort that survives at some future point in time (see Tan et al. (2015) for a detailed overview). However, in contrast to all existing literature, we propose an equilibrium approach, which we discover to be particularly useful in the current state of the life market, where transactions are made over the counter between parties and where one cannot assume the existence of a liquid market.

The pricing of any mortality linked derivative security begins with the choice of a mortality model. We position ourselves in the context of dynamic mortality models; in particular we use continuous-time mortality models of intensity type. In the existing literature on this topic, in the one-cohort case, one particularly notes works by Milevsky and Promislow (2001), Dahl (2004), Biffis (2005) and Luciano and Vigna (2008). Apart from recent models by Jalen and Mamon (2009), Qian et al. (2010) and Deelstra et al. (2015), most of cohort-based continuous time models assume independence of financial and demographic risk factors. Although this assumption has practical advantages, and there is an array of situations where this assumption is warranted, we will refrain from making it. The reason for this is the emerging body of empirical research, exemplified in works by Ang and Maddaloni (2003), Favero et al. (2011) , Maurer (2011b) and Dacorogna and Cadena (2015), all of which suggest a connection between long-run demographic trends and financial markets. Therefore, it appears natural to consider stochastic intensity models which are correlated to some extent to financial securities. In particular, we consider a model for the risky assets in which the market price of risk depends on the mortality intensity.

The paper is organized as follows. In Section 2, we introduce the equilibrium pricing approaches from the standpoint of relevant recent literature. Next, in Section 3 we introduce our model, and in Section 4, we present the equilibrium prices. In Section 5, we address the issue of market completeness, and, finally, we conclude with Section 6.

2 Modeling approach

In this paper we investigate the equilibrium pricing of a longevity bond associated with certain cohort whose stochastic mortality intensity affects the market participants' cashflows. There is one risk-free asset and one risky asset in the financial market. The risky asset price is exogenously given, i.e., it was priced in another market, and this position ourselves in a partial equilibrium model. The market participants are economic agents who decide their optimal trading strategies based on their preferences which are described by convex dynamic risk measures generated by the solutions of backward stochastic differential equation (BSDE). The agents' cashflow is affected by the stochastic mortality of certain cohort but the mortality intensity itself is non-tradable. Therefore, we introduce longevity bond market that can be utilized to hedge the stochastic mortality. The price of the longevity bond is determined in equilibrium by market clearing of longevity bond market. By using Clark-Haussmann formula, we prove that the price process of longevity bond has non-zero volatility with respect to the randomness driving the stochastic mortality. This means that our longevity bond market can be used to hedge the stochastic mortality and leads us to a market completion.

This work is an extension and an application of the frameworks developed by Horst and Müller (2007) and Horst et al. (2010). Horst and Müller (2007) propose an equilibrium pricing method for financial security written on non-tradable underlying. In their model, the non-tradable underlying has idiosyncratic risk source and it is independent of the tradable stock in the market. The preference of the agents are characterized by expected utility with exponential utility function. Horst et al. (2010) focus on problems with more general preferences than Horst and Müller (2007), i.e., they consider risk preferences generated by convex dynamic risk measures. In Horst et al. (2010), the stock price and non-tradable underlying may have dependency through the drift of non-tradable underlying which is an adapted stochastic process. However, the volatility of non-tradable underlying is assumed to be constant. Moreover, it is also assumed in Horst et al. (2010) that the payoff of financial security written on the non-tradable underlying only depends on the terminal value of stock and non-tradable underlying. While Horst and Müller (2007) and Horst et al. (2010) consider climate risk and weather derivative to explain/motivate their models, we apply a similar equilibrium pricing framework to longevity bond pricing. However, we consider a more general dependency structure between the risky asset and the mortality intensity than that in Horst et al. (2010). Moreover, we can prove that the longevity bond completes the market even if the payoff of longevity bond is path dependent. In contrast to our continuous time model, there are studies that consider discrete time models such as Cocco and Gomes (2012), which examine the benefit from investing in financial securities that are designed to hedge longevity risk.

3 Model Setup

3.1 Risk sources and assets

Let $W_t = (W_t^S, W_t^R)^\top$ be a standard two-dimensional Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, where $W_t^S$and $W_t^R$are independent 1-dimensional Brownian motions. $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the filtration generated by W_t. For simplicity, it is assumed that the risk-free rate is zero under $\mathbb{P}$. The risky asset has the following dynamics

$$\frac{dS_t}{S_t} = \mu_t^S dt + \sigma_t^S dW_t^S \quad \text{where} \quad S_0 > 0, \tag{1}$$

where μ_t^S and σ_t^S are $\mathcal{F}_t$-adapted processes with $\sigma_t^S > 0$ for all $t \geq 0$ almost surely (a.s.).

To describe mortality experience of a predetermined cohort (see Biffis (2005) and Luciano and Vigna (2008)) we place ourself in the Cox process setting (see Lando (2009)). Here, intuitively, the first jump time of a Poisson process with stochastic (mortality) intensity represents the time of death of life belonging to a given cohort. Hence, on the introduced probability space, let us consider a predictable process R_t having the following dynamics

$$dR_t = \mu^R R_t dt + b_1 dW_t^S + b_2 dW_t^R, \quad \text{where} \quad \mu^R \in \mathbb{R}^+,\ b_1 \in \mathbb{R}, b_2 > 0$$

which represents the mortality intensity of an individual belonging to a predetermined cohort (generation).

In our model, two Brownian motions W_t^S and W_t^R represent the financial risk and the systemic mortality risk[2] respectively. We assume that the risky asset is freely traded, in the market, whereas the mortality intensity is non-tradable. Note that b_2 is not zero so that there exists systemic mortality risk not spanned by the risky asset. Since the mortality intensity itself is not traded, the market is incomplete if there are no additional asset that can be utilized to hedge the systemic mortality risk.

In this paper we will investigate partial equilibrium approach to pricing of zero coupon longevity bond written on specific cohort and a given duration. Though potentially illiquid instrument due to strong tendency of buy and hold strategy of market participants (see Blake et al. (2006)) its pricing via partial equilibrium approach is appropiate to address this situation. Also, as zero coupon longevity bonds can be written of different cohort and have various durations, they represent an ideal building blocks for pricing more complex mortality linked products especially in situations of over the counter transactions.

In the following H^D is the payoff of a zero coupon longevity bond at time T for notional of size C depends only on the realized survival rate. We assume that

$$H^D = G(p(0,T)),$$

[2]Throughout this paper, it is assumed that all idiosyncratic mortality risks are diversified by existence of appropriate number of large annuity portfolios.

where

$$p(t,T) = e^{-\int_t^T R_s ds} \tag{2}$$

and the function G is a monotone increasing (or decreasing) function, which is not a constant function.[3]

We proceed to pricing of this longevity derivative by finding the appropriate market-price-of-risk (MPR) process. Let $\{\theta_t\}_{0\le t\le T} = \{(\theta_t^S, \theta_t^R)\}_{0\le t\le T}$ be a two-dimensional predictable stochastic process that makes $\{\zeta_t^\theta\}_{0\le t\le T}$ be a uniformly integrable martingale, where ζ_t^θ is defined by

$$\zeta_t^\theta = \exp\left(-\frac{1}{2}\int_0^t \|\theta_s\|^2 ds - \int_0^t \theta_s dW_s\right), \tag{3}$$

and $\|\cdot\|$ is Euclidean norm. Then, we can define $\mathbb{P}^\theta$ with density ζ_t^θ, which is a probability measure equivalent to $\mathbb{P}$, $\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \zeta_T^\theta$, and

$$W_t^\theta = (W_t^{\theta,S}, W_t^{\theta,R}) = \left(W_t^S + \int_0^t \theta_s^S ds, W_t^R + \int_0^t \theta_s^R ds\right)$$

becomes the standard two-dimensional Brownian motion under $\mathbb{P}^\theta$. Then, the pricing of the longevity derivative is equivalent to finding the market-price-of-risk (MPR) process $\{\theta_t\}_{0\le t\le T} = \{(\theta_t^S, \theta_t^R)\}_{0\le t\le T}$ that allows $\mathbb{P}^\theta$ to be the equivalent martingale measure. Since the risky asset can be freely traded, it is obvious that $\{\theta_t^S\}_{0\le t\le T}$, which is referred to as the MPR of financial risk, is

$$\theta_t^S = \frac{\mu_t^S}{\sigma_t^S}.$$

However, $\{\theta_t^R\}_{0\le t\le T}$, which is referred to as the MPR of systemic mortality risk, is not trivial because the mortality intensity itself is not tradable. It will be determined by the equilibrium condition of longevity derivative market that consists of several market participants who optimally decide their portfolios. More details will follow after we introduce the preference and the optimization problems of agents who participate in longevity derivative market.

[3]More precisely we assume that $x \to G(x)$ is differentiable except at finite set of points $\{y_1, y_2, \dots, y_n\}$ and that $x \to G(x)$ is strictly monotone on some interval in the domain. One payoff profile which falls in this category is given in Section 5. The set $\mathcal{N} = \{\int_0^T R_s ds = \frac{1}{x_i}, i = 1, 2, \dots, n\}$ is a $\mathbb{P}-$null set and therefore can be omitted while doing the analysis.

If we find the equilibrium MPR process θ_t^{*R} of systemic mortality risk, then the price process of longevity bond for given $\theta_t^* = (\theta_t^S, \theta_t^{*R})$ with payoff H^D at time T is

$$B_t^{\theta^*} = \mathbb{E}^{\theta^*}[H^D|\mathcal{F}_t] = \mathbb{E}\Big[\frac{1}{\zeta_t^{\theta^*}}\zeta_T^{\theta^*}H^D\Big|\mathcal{F}_t\Big], \tag{4}$$

where $\mathbb{E}^{\theta^*}[\cdot]$ denotes the expectation under probability measure $\mathbb{P}^{\theta^*}$. By martingale representation theorem, B_t^θ can be written as

$$\begin{aligned} B_t^{\theta^*} =& H^D - \int_t^T \kappa_s^{\theta^*,S} dW_s^{\theta^*,S} - \int_t^T \kappa_s^{\theta^*,R} dW_s^{\theta^*,R} \\ =& H^D - \int_t^T \kappa_s^{\theta^*,S}(dW_s^S + \theta_s^S ds) - \int_t^T \kappa_s^{\theta,R}(dW_s^R + \theta_s^{*R} ds), \end{aligned} \tag{5}$$

where $\kappa_s^{\theta^*,S}$ and $\kappa_s^{\theta^*,R}$ are adapted volatility processes.

Before closing this subsection we want to make the following assumption:

Assumption 1. *We assume that θ_t^S is a function of t, S_t, and R_t, that is,*

$$\theta_t^S = F(t, S_t, R_t)$$

for a uniformly bounded function $F(t,\cdot,\cdot)$, with uniformly bounded first and second derivatives. Thus $(\theta_t^S)_{0\le t\le T}$ is an $\mathcal{F}_t$-adapted Markov process.

Remark 1. *Maurer (2011a) considers an equilibrium pricing model which yields that the endogeneous market price of risk is (under certain conditions) higher during periods characterised by a high birth rate (baby boom) and low mortality than in times of a low birth rate and a high death rate. They conclude that "demographic changes appear to explain substantial parts of the time variation in the real interest rate, the market price of risk and the equity premium. Due to demographic uncertainty the conditional volatility of stock returns is stochastically changing over time and the unconditional volatility of asset returns is substantially larger than the unconditional variation in aggregate consumption growth."*

3.2 Agent's optimization

Now we introduce the optimization problem of agents who participate in the longevity derivatives market. Let $I \in \{i_1, i_2, \ldots, i_N\}$ be the set of agents in the market. Then the gain of an agent $i \in I$ from trading the risky asset and the longevity derivative up to time t under $\mathbb{P}^\theta$ using self-financing strategy $\{\pi_s^i = (\pi_{1,s}^i, \pi_{2,s}^i)\}_{0\le s\le T}$ with zero initial capital is given by

$$V_t^{i,\theta}(\pi^i) = \int_0^t \pi_{1,s}^i dS_s + \int_0^t \pi_{2,s}^i dB_s^\theta.$$

Let us assume that the agent i's net cash flow up to time T without trading is

$$H^i = H^i(S, R),$$

which may depend on the path of risky asset $S = \{S_s\}_{0 \leq s \leq T}$ and the path of mortality intensity process $R = \{R\}_{0 \leq s \leq T}$. At time T, by following strategy π^i, the agent i will have

$$H^i + V_T^{i,\theta}(\pi^i).$$

It is assumed that the agent i's optimal decision is made based on a risk process Y_t^i that satisfies the following backward stochastic differential equation (BSDE):

$$\begin{aligned} -dY_t^i(\pi^i) &= g^i(t, Z_t^i)dt - Z_t^i dW_t \\ Y_T^i &= -(H^i + V_T^{i,\theta}(\pi^i)) \end{aligned}$$

or equivalently,

$$Y_t^i(\pi^i) = -(H^i + V_T^{i,\theta}(\pi^i)) + \int_t^T g^i(s, Z_s^i)ds - \int_t^T Z_s^i dW_s, \tag{6}$$

where $g^i(t, z)$ is the driver of the BSDE and $Z_s^i = (Z_{1,s}^i, Z_{2,s}^i)$ is a two-dimensional progressively measurable process. The preference of the agent i is determined by the driver $g^i(t, z)$ of the BSDE. We assume that the driver of agent i is given by

$$g^i(t, z) = \frac{1}{2\gamma_i}\|z\|^2 \tag{7}$$

which corresponds to the dynamic entropic risk measure, equivalent to exponential utility. In this case, γ_i is the risk tolerance parameter of agent i. By following Horst et al. (2010), we define the set of admissible trading strategies as follows

Definition 1 (Horst et al. (2010))**.** *Let $\mathcal{H}_T^p(\mathbb{R}^m, \mathbb{P})$ having $m \in \mathbb{N}$, be the space of all progressively measurable processes $\{X_t\}_{0 \leq t \leq T}$ with values in $\mathbb{R}^m$ such that $\mathbb{E}\left[(\int_0^T \|X_s\|^2 ds)^{p/2}\right] < \infty$ where $p \in \{1, 2\}$. Then, if the BSDE* (6) *has a unique solution $(Y_t^i, Z_t^i) \in \mathcal{H}_T^1(\mathbb{R}, \mathbb{P}) \times \mathcal{H}_T^2(\mathbb{R}^2, \mathbb{P})$ and $\mathbb{E}[\langle V^{i,\theta}(\pi^i)\rangle_T] < \infty$, where $\langle\cdot\rangle_t$ denotes the quadratic variation, a strategy π^i is admissible with respect to the given MPR θ. For given θ, $\mathcal{S}^\theta$ is the set of all admissible strategies with respect to θ.*

Then, the optimization problem of the agent a at time t is to find the optimal trading strategy $\bar{\pi}^i$ defined as follows

$$\{\bar{\pi}_s^i\}_{t \leq s \leq T} := \underset{\pi^i \in \mathcal{S}^\theta}{\operatorname{argmin}}\, Y_t^i(\pi^i). \tag{8}$$

Therefore, the agent optimally chooses her trading strategies to minimize the risk process Y_t^i.

Note that, at time t, $V_t^{i,\theta}(\pi^i)$ is already determined and not affected by the trading strategies π_s for $s \geq t$. Thus, if we define a residual risk process[4] $\bar{Y}_t^i(\pi^i) := Y_t^i(\pi^i) + V_t^{i,\theta}(\pi^i)$, and consider the optimization of residual risk $\bar{Y}_t^i$ instead of the original risk process Y_t^i, the residual risk minimization problem gives us the same optimal trading strategy that can be obtained from the original risk minimization problem in (8). In short

$$\{\bar{\pi}_s^i\}_{s\geq t} = \underset{\pi^i \in \mathcal{S}^\theta}{\operatorname{argmin}}\, Y_t^i(\pi^i) = \underset{\pi^i \in \mathcal{S}^\theta}{\operatorname{argmin}}\, \bar{Y}_t^i(\pi^i). \tag{9}$$

From (1) and (5), we have

$$\begin{aligned}
V_T^{i,\theta}(\pi^i) - V_t^{i,\theta}(\pi^i) &= \int_t^T \pi_{1,s}^i dS_s + \int_t^T \pi_{2,s}^i dB_s^\theta \\
&= \int_t^T \left(\pi_{1,s}^i \sigma_s^S \theta_s^S S_s + \pi_{2,s}^i (\kappa_s^{\theta,S}\theta_s^S + \kappa_s^{\theta,R}\theta_s^R)\right) ds \\
&+ \int_t^T \left(\pi_{1,s}^i \sigma_s^S S_s + \pi_{2,s}^i \kappa_s^{\theta,S}\right) dW_s^S + \int_t^T \pi_{2,s}^i \kappa_s^{\theta,R} dW_s^R,
\end{aligned}$$

and thus, it follows that

$$\begin{aligned}
\bar{Y}_t^i(\pi^i) =& Y_t^i(\pi^i) + V_t^{i,\theta}(\pi^i) \\
=& -H^i - (V_T^{i,\theta}(\pi^i) - V_t^{i,\theta}(\pi^i)) + \int_t^T g^i(s, Z_s^i)ds - \int_t^T Z_s^i dW_s \\
=& -H^i + \int_t^T \left(g^i(s, Z_s^i) - \pi_{1,s}^i \sigma_s^S \theta_s^S S_s - \pi_{2,s}^i (\kappa_s^{\theta,S}\theta_s^S + \kappa_s^{\theta,R}\theta_s^R)\right) ds \\
&- \int_t^T \left(Z_{1,s}^i + \pi_{1,s}^i \sigma_s^S S_s + \pi_{2,s}^i \kappa_s^{\theta,S}\right) dW_s^S - \int_t^T \left(Z_{2,s}^i + \pi_{2,s}^i \kappa_s^{\theta,R}\right) dW_s^R, \\
=& -H^i + \int_t^T \bar{g}^i(s, \pi_s^i, \bar{Z}_s^i)ds - \int_t^T \bar{Z}_s^i dW_s,
\end{aligned}$$

with the change of variable

$$\bar{Z}_s^i = (\bar{Z}_{1,s}^i, \bar{Z}_{2,s}^i) = (Z_{1,s}^i + \pi_{1,s}^i \sigma_s^S S_s + \pi_{2,s}^i \kappa_s^{\theta,S}, Z_{1,s}^i + \pi_{2,s}^i \kappa_s^{\theta,R})$$

[4] $\bar{Y}_t^i$ is called the residual risk because it is the risk Y_t^i less $V_t^{i,\theta}$ that is the past trading gain up to time t. See Horst et al. (2010).

and the new driver

$$
\begin{aligned}
\bar{g}^i(s,\pi,z) &= \bar{g}^i(s,(\pi_1,\pi_2),(z_1,z_2)) \\
&= g^i\left(s,(z_1-\pi_1\sigma_s^S S_s-\pi_2\kappa_s^{\theta,S},z_2-\pi_2\kappa_s^{\theta,R})\right)-\pi_1\sigma_s^S\theta_s^S S_s-\pi_2(\kappa_s^{\theta,S}\theta_s^S+\kappa_s^{\theta,R}\theta_s^R) \\
&= \frac{1}{2\gamma_i}\left\{(z_1-\pi_1\sigma_s^S S_s-\pi_2\kappa_s^{\theta,S})^2+(z_2-\pi_2\kappa_s^{\theta,R})^2\right\}-\pi_1\sigma_s^S\theta_s^S S_s-\pi_2(\kappa_s^{\theta,S}\theta_s^S+\kappa_s^{\theta,R}\theta_s^R).
\end{aligned}
\tag{10}
$$

By comparison principle, the optimal strategy is the strategy that minimizes the driver $\bar{g}^i(s,\pi,z)$. Thus, a candidate of optimal strategy can be derived from the first order conditions (FOC)

$$
\frac{\partial\bar{g}^i(s,\pi_s^i,\bar{Z}^i)}{\partial\pi_{1,s}^i}=0 \quad \text{and} \quad \frac{\partial\bar{g}^i(s,\pi_s^i,\bar{Z}^i)}{\partial\pi_{2,s}^i}=0, \tag{11}
$$

where $\bar{g}^i(s,\pi,z)$ is given by (10). Since it can be shown that the candidate strategy obtained from the FOC (11) is admissible[5], the candidate strategy is indeed the optimal strategy. Therefore, the first order conditions in (11) allow us the optimal strategy as follows:

$$
\bar{\pi}_{1,s}^i=\frac{\bar{Z}_{1,s}^i+\gamma_i\theta_s^S-\bar{\pi}_{2,s}^i\kappa_s^{\theta,S}}{\sigma_s^S S_s} \quad \text{and} \quad \bar{\pi}_{2,s}^i=\frac{\bar{Z}_{2,s}^i+\gamma_i\theta_s^R}{\kappa_s^{\theta,R}} \tag{12}
$$

4 Equilibrium Pricing

If the optimal strategies of agents in the market for given MPR are derived, then the equilibrium MPR is determined by the market clearing condition for the longevity derivative market. Let $n \geq 0$ be the units of longevity derivative in the market available for trading. Then, we define the equilibrium MPR as follow:

Definition 2. *Let $\{\bar{\pi}_t^i\}_{0\leq t\leq T}=\{(\bar{\pi}_{1,t}^i,\bar{\pi}_{2,t}^i)\}_{0\leq t\leq T}$ be the admissible optimal strategy for agent i's optimization in (9). Then, $\{\theta_t\}_{0\leq t\leq T}=\{(\theta_t^S,\theta_t^R)\}_{0\leq t\leq T}$ is called an equilibrium MPR if the MPR of systemic mortality risk $\{\theta_t^R\}_{0\leq t\leq T}$ satisfies the market clearing condition*

$$
\sum_{i\in I}\bar{\pi}_{2,t}^i\equiv n, \quad a.s. \tag{13}
$$

for all $t\in[0,T]$.

[5]This is guaranteed by the assumption of driver in (7) and the results of Kobylansky (2000) about the existence and uniqueness of solution of BSDE with a driver that has quadratic growth.

Instead of using our model with the equilibrium condition (13) directly, we introduce an equivalent model that has zero net supply of longevity derivatives, which allows us same equilibrium MPR along with more convenient analysis.[6] Consider an equivalent model satisfying following two conditions

1) The optimization problem of agent $i \in I$ is given as follow

$$\{\tilde{\pi}_s^i\}_{s \geq t} = \underset{\tilde{\pi}^i \in \mathcal{S}^\theta}{\operatorname{argmin}} \check{Y}_t^i(\tilde{\pi}^i), \text{ where} \tag{14}$$

$$\begin{aligned}\check{Y}_t^i(\tilde{\pi}^i) = &-(H^i + \nu^i H^D) + \int_t^T \left(g^i(s, Z_s^i) - \tilde{\pi}_{1,s}^i \sigma_s^S \theta_s^S S_s - \tilde{\pi}_{2,s}^i (\kappa_s^{\theta,} \theta_s^S + \kappa_s^{\theta,R} \theta_s^R) \right) ds \\ &- \int_t^T \left(Z_{1,s}^i + \tilde{\pi}_{1,s}^i \sigma_s^S S_s + \tilde{\pi}_{2,s}^i \kappa_s^{\theta,S} \right) dW_s^S - \int_t^T \left(Z_{2,s}^i + \tilde{\pi}_{2,s}^i \kappa_s^{\theta,R} \right) dW_s^R\end{aligned}$$

and ν^i satisfies $\sum_{i \in I} \nu^i = n$.

2) For all $t \in [0, T]$, the market clearing condition is

$$\sum_{i \in I} \tilde{\pi}_{2,t}^i \equiv 0 \quad \text{a.s..} \tag{15}$$

In this equivalent model, the agent i has additional endowment $\nu^i H^D$ compared to the original model, where H^D is the payoff of the longevity derivative. Therefore, the optimal demand for the longevity derivative decreases exactly by ν^i. Since the sum of ν^i of the agents in the market is n, the corresponding market clearing condition that gives same MPR to the original model is (15), instead of (13).

The lemma below summarizes the above argument.

Lemma 1. *Let $\bar{\pi}_s^i = (\bar{\pi}_{1,s}^i, \bar{\pi}_{2,s}^i)$ be the optimal strategy for the original model that is defined by* (8)*. Then, $(\bar{\pi}_{1,s}^i, \bar{\pi}_{2,s}^i - \nu^i)$ is the optimal strategy for the new model, that is,*

$$\tilde{\pi}_s^i = (\tilde{\pi}_{1,s}^i, \tilde{\pi}_{2,s}^i) = (\bar{\pi}_{1,s}^i, \bar{\pi}_{2,s}^i - \nu^i), \tag{16}$$

where $\tilde{\pi}_s^i$ is defined by (14)*. On the other hand, if $\tilde{\pi}_s^i = (\tilde{\pi}_{1,s}^i, \tilde{\pi}_{2,s}^i)$ is the optimal strategy for the new model that is defined by* (14)*, then $(\tilde{\pi}_{1,s}^i, \tilde{\pi}_{2,s}^i + \nu^i)$ is optimal for the original model and this means that*

$$\bar{\pi}_s^i = (\bar{\pi}_{1,s}^i, \bar{\pi}_{2,s}^i) = (\tilde{\pi}_{1,s}^i, \tilde{\pi}_{2,s}^i + \nu^i). \tag{17}$$

[6] This approach is already introduced and used in Horst et al. (2010).

Proof. See Appendix A. □

In the original model, the equilibrium MPR θ^{*R} is determined by the market clearing condition in (13). By Lemma 1, the market clearing condition (13) for the original model is equivalent to

$$\sum_{i\in I}\tilde{\pi}^i_{2,t}=\sum_{i\in I}(\bar{\pi}^i_{2,t}-\nu^i)=\sum_{i\in I}\bar{\pi}^i_{2,t}-\sum_{i\in I}\nu^i=n-n=0$$

because $\sum_{i\in I}\nu^i = n$. Thus, Lemma 1 implies that, in the new model, by imposing a new equilibrium condition (15) instead of (13), we can obtain the equilibrium MPR of systemic mortality risk θ^{*R} that coincides with the equilibrium MPR of systemic mortality risk for the original model.

Now we derive the optimal strategy of agent i in the equivalent model, which is the solution of the optimization in (14). Note that

$$\begin{aligned}\check{Y}^i_t(\check{\pi}^i)=&-(H^i+\nu^iH^D)+\int_t^T\Big(g^i(s,Z^i_s)-\check{\pi}^i_{1,s}\sigma^S_s\theta^S_sS_s-\check{\pi}^i_{2,s}(\kappa^{\theta,S}_s\theta^S_s+\kappa^{\theta,R}_s\theta^R_s)\Big)\,ds\\&-\int_t^T\Big(Z^i_{1,s}+\check{\pi}^i_{1,s}\sigma^S_sS_s+\check{\pi}^i_{2,s}\kappa^{\theta,S}_s\Big)\,dW^S_s-\int_t^T\Big(Z^i_{2,s}+\check{\pi}^i_{2,s}\kappa^{\theta,R}_s\Big)\,dW^R_s\\=&-(H^i+\nu^iH^D)+\int_t^T\check{g}^i(s,\check{\pi}^i_s,\check{Z}^i_s)ds-\int_t^T\check{Z}^i_sdW_s,\end{aligned}$$

with change of variable

$$\check{Z}^i_s=(\check{Z}^i_{1,s},\check{Z}^i_{2,s})=(Z^i_{1,s}+\check{\pi}^i_{1,s}\sigma^S_sS_s+\check{\pi}^i_{2,s}\kappa^{\theta,S}_s,\,Z^i_{2,s}+\check{\pi}^i_{2,s}\kappa^{\theta,R}_s)$$

and the new driver

$$\check{g}^i(s,\check{\pi},\check{z})=\frac{1}{2\gamma_i}\Big\{(\check{z}_1-\check{\pi}_1\sigma^S_sS_s-\check{\pi}_2\kappa^{\theta,S}_s)^2+(\check{z}_2-\check{\pi}_2\kappa^{\theta,R}_s)^2\Big\}-\check{\pi}_1\sigma^S_s\theta^S_sS_s-\check{\pi}_2(\kappa^{\theta,S}_s\theta^S_s+\kappa^{\theta,R}_s\theta^R_s). \tag{18}$$

It is straightforward that the optimal strategy for the new model can be derived by following similar procedures we used for the optimization of original model in Section 3.2. Indeed, the optimal strategy for the new model is

$$\check{\pi}^i_{1,s}=\frac{\check{Z}^i_{1,s}+\gamma_i\theta^S_s-\check{\pi}^i_{2,s}\kappa^{\theta,S}_s}{\sigma^S_sS_s}\quad\text{and}\quad\check{\pi}^i_{2,s}=\frac{\check{Z}^i_{2,s}+\gamma_i\theta^R_s}{\kappa^{\theta,R}_s}. \tag{19}$$

By substituting the optimal strategy in (19) into the BSDE, we have the BSDE for agent $i\in I$ as follows:

$$\check{Y}^i_t=-(H^i+\nu^iH^D)+\int_t^T\Big\{-\frac{\gamma_i}{2}(\theta^S_s)^2-\check{Z}^i_{1,s}\theta^S_s-\frac{\gamma_i}{2}(\theta^R_s)^2-\check{Z}^i_{2,s}\theta^R_s\Big\}ds-\int_t^T\check{Z}^i_sdW_s. \tag{20}$$

In this case, the market clearing condition (15) becomes

$$\sum_{i\in I}\frac{\check{Z}^i_{2,s}+\gamma_i\theta^{*R}_s}{\kappa^{\theta^*,R}_s}=0,$$

where θ^{*R} is the equilibrium MPR of systemic mortality risk, and it is equivalent to[7]

$$\theta^{*R}_s=-\frac{\sum_{i\in I}\check{Z}^i_{2,s}}{\sum_{i\in I}\gamma_i}. \tag{21}$$

Although we can represent the equilibrium MPR θ^{*R} by (21), it is complicated to obtain the solution by using this approach directly because we need to know θ^{*R} to find the solution $(\check{Y}^i_t,\check{Z}^i_t)$ of agent i's BSDE, while θ^{*R} can be found when we solve the optimization problems of all agents in the market. Instead, let us define $\check{Y}^I_t$, $\check{Z}^I_t$, and γ_I as follows

$$\check{Y}^I_t=\sum_{i\in I}\check{Y}^i_t,\quad \check{Z}^I_t=\sum_{i\in I}\check{Z}^i_t,\quad\text{and}\quad \gamma_I=\sum_{i\in I}\gamma_i.$$

From (20), we have

$$\begin{aligned}\check{Y}^I_t=&-(\sum_{i\in I}H^i+\sum_{i\in I}\nu^iH^D)\\&+\int_t^T\Big\{-\frac{1}{2}\sum_{i\in I}\gamma_i(\theta^S_s)^2-\sum_{i\in I}\check{Z}^i_{1,s}\theta^S_s-\frac{1}{2}\sum_{i\in I}\gamma_i(\theta^R_s)^2-\sum_{i\in I}\check{Z}^i_{2,s}\theta^R_s\Big\}ds-\int_t^T\sum_{i\in I}\check{Z}^i_sdW_s\\=&-(\sum_{i\in I}H^i+nH^D)+\int_t^T\Big\{-\frac{\gamma_I}{2}(\theta^S_s)^2-\check{Z}^I_{1,s}\theta^S_s-\frac{\gamma_I}{2}(\theta^R_s)^2-\check{Z}^I_{2,s}\theta^R_s\Big\}ds-\int_t^T\check{Z}^I_sdW_s\end{aligned} \tag{22}$$

and the equilibrium MPR of systemic mortality risk in (21) becomes

$$\theta^{*R}_s=-\frac{\check{Z}^I_{2,s}}{\gamma_I}. \tag{23}$$

By substituting (23) into the BSDE (22), we can obtain the following BSDE

$$\check{Y}^I_t=-(\sum_{i\in I}H^i+nH^D)+\int_t^T\Big\{-\frac{\gamma_I}{2}(\theta^S_s)^2-\check{Z}^I_{1,s}\theta^S_s+\frac{(\check{Z}^I_{2,s})^2}{2\gamma_I}\Big\}ds-\int_t^T\check{Z}^I_sdW_s. \tag{24}$$

Now, we only need to find the solution $(\check{Y}^I_t,\check{Z}^I_t)$ of a single BSDE (24) to derive the equilibrium MPR θ^{*R}. Moreover, by multiplying the both sides of the BSDE (24) by $-1/\gamma_I$ and redefining

[7]This step demonstrates why we introduce the new model with zero net supply of longevity derivatives.

the processes as $Y_t^I = \check{Y}_t^I$, $Z_{1,t}^I = -\frac{1}{\gamma_I}\check{Z}_{1,t}^I$, and $Z_{2,t}^I = -\frac{1}{\gamma_I}\check{Z}_{2,t}^I$, we can derive the following BSDE, of which γ_I does not appears in the driver.

$$Y_t^I = \frac{(\sum_{i\in I} H^i + nH^D)}{\gamma_I} + \int_t^T \Big\{\frac{1}{2}(\theta_s^S)^2 - Z_{1,s}^I\theta_s^S - \frac{(Z_{2,s}^I)^2}{2}\Big\}ds - \int_t^T Z_s^I dW_s. \tag{25}$$

In this case, if we find the solution (Y_t^I, Z_t^I) to the BSDE (25), then the equilibrium MPR θ^{*R} is given by

$$\theta_s^{*R} = Z_{2,s}^I. \tag{26}$$

Remark 2. *Our approach resembles Horst and Müller (2007) in the way that it does not utilize the representative agent. The approach of Horst et al. (2010), based on the representative agent, covers more cases of risk preferences, but this comes with some extra technicalities. That is to say the approach of our paper, in the context of the specific risk preferences that we consider, is simpler than the approach of Horst et al. (2010). However, the two approaches are equivalent, because they lead to the same result.*

Since we already obtained the equilibrium MPR θ^{*R}, we can find the longevity bond price process $B_t^{\theta^*}$ for the given equilibrium MPR process by the equation (4). It turns out that the longevity bond indeed completes the market, that is, $\kappa^{\theta^*,R}$ is not zero a.s.. Let us summarize our findings in the following Theorem which is the main result of our paper.

Theorem 1. *Let* (Y_s^I, Z_s^I) *be the solution to the BSDE* (25)*. Then, the process* θ^{*R} *of* (26)*, where* $Z_{2,s}^I$ *is the second component of the* Z_s^I*, is an equilibrium MPR of systemic mortality risk, and the price of the longevity bond is given by* (4)*. Moreover, the longevity bond completes the market.*

Proof. See Appendix B. □

5 Numerical Results

We provide numerical examples in this section. The outline of the approach is as follows: first, we use discretization and simulations to obtain samples of H^D and sample paths of (Y_t^I, Z_t^I) that corresponds to the BSDE (25), then the equilibrium MPR of systemic mortality risk θ^{*R} is just the second component of Z_t^I as we have already seen in (26). By using the equilibrium

MPR to construct the sample paths of $\zeta_t^{\theta^*}$, we can compute the price of the longevity bond by using the equation (4), which requires numerical integration.

Throughout this section, for simplicity of exposition, let us assume that $\mu_t^S \equiv \mu^S$ and $\sigma_t^S \equiv \sigma^S$, where μ^S and σ^S are positive constants, then it follows that $\theta_t^S = \mu^S/\sigma^S$ is also a positive constant.

5.1 Numerical scheme

First, let us divide the time horizon $[0, T]$ into N time intervals: $0 = t_0 < t_1 < \cdots < t_N = T$, where $t_i = i\Delta t$, for $i = 0, 1, 2, \ldots, N$ with $\Delta t = T/N$. If we generate M sample paths of $W_{t_m}^S$ and $W_{t_m}^R$, for $m = 1, 2, \ldots, N$, then we can obtain the sample paths of S_{t_m}, R_{t_m}, and $p(0, t_m)$ as follows:

$$S_{t_m} = S_0 \exp\Big(\mu^S - \frac{(\sigma^S)^2}{2} t_m + \sigma^S W_{t_m}^S\Big),$$

$$R_{t_m} = R_{t_{m-1}}(1 + \mu^R \Delta t) + b_1 \Delta W_{t_{m-1}}^S + b_2 \Delta W_{t_{m-1}}^R,$$

$$p_{t_m} = p_{t_{m-1}} \exp(-R_{t_{m-1}} \Delta t),$$

for $m = 1, 2, \ldots, N$, where $\Delta W_{t_m}^S = (W_{t_{m+1}}^S - W_{t_m}^S)$ and $\Delta W_{t_m}^R = (W_{t_{m+1}}^R - W_{t_m}^R)$. Then, we have M random samples of H^i, for $i \in I$, and H^D because they depend on the paths of S and R.

Since the equilibrium MPR of systemic mortality risk θ_t^{*R} is indeed $Z_{2,t}^I$, we need sample paths of $Z_{2,t}^I$. From the BSDE (25), we have

$$Y_{t_m}^I = Y_{t_{m+1}}^I + \int_{t_m}^{t_{m+1}} \Big\{\frac{1}{2}(\theta^S)^2 - Z_{1,s}^I \theta^S - \frac{(Z_{2,s}^I)^2}{2}\Big\} ds - \int_{t_m}^{t_{m+1}} Z_s^I dW_s, \tag{27}$$

for $m = 0, 1, 2, \ldots, N-1$. By multiplying the both sides of (27) by $\Delta W_{t_m}^S$ and taking conditional expectation $\mathbb{E}_{t_m}[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_{t_m}]$, we have

$$\begin{aligned}
0 =& \mathbb{E}_{t_m}\Big[\Delta W_{t_m}^S Y_{t_m}^I\Big] \\
=& \mathbb{E}_{t_m}\Big[\Delta W_{t_m}^S \Big(Y_{t_{m+1}}^I + \int_{t_m}^{t_{m+1}} \Big\{\frac{1}{2}(\theta^S)^2 - Z_{1,s}^I \theta^S - \frac{(Z_{2,s}^I)^2}{2}\Big\} ds - \int_{t_m}^{t_{m+1}} Z_s^I dW_s\Big)\Big] \\
=& \mathbb{E}_{t_m}\Big[\Delta W_{t_m}^S Y_{t_{m+1}}^I\Big] + \int_{t_m}^{t_{m+1}} \mathbb{E}_{t_m}\Big[(W_s^S - W_{t_m}^S)\Big\{\frac{1}{2}(\theta^S)^2 - Z_{1,s}^I \theta^S - \frac{(Z_{2,s}^I)^2}{2}\Big\}\Big] ds - \int_{t_m}^{t_{m+1}} \mathbb{E}_{t_m}\Big[Z_{1,s}^I\Big] ds \\
\approx& \mathbb{E}_{t_m}\Big[\Delta W_{t_m}^S Y_{t_{m+1}}^I\Big] - Z_{1,t_m}^I \Delta t,
\end{aligned} \tag{28}$$

where the last line of (28) is obtained by using rectangular method to approximate the integrals. It follows from (28) that

$$Z^I_{1,t_m} \approx \frac{1}{\Delta t}\mathbb{E}_{t_m}\Big[\Delta W^S_{t_m} Y^I_{t_{m+1}}\Big]. \tag{29}$$

Similarly, by multiplying the both sides of (27) by $\Delta W^R_{t_m}$ and taking conditional expectation $\mathbb{E}_{t_m}[\cdot]$, and also using rectangular method to approximate the integrals, we can derive

$$Z^I_{2,t_m} \approx \frac{1}{\Delta t}\mathbb{E}_{t_m}\Big[\Delta W^R_{t_m} Y^I_{t_{m+1}}\Big]. \tag{30}$$

By taking conditional expectation $\mathbb{E}_{t_m}[\cdot]$ directly on both sides of (27), we can derive

$$\begin{aligned} Y^I_{t_m} &= \mathbb{E}_{t_m}\Big[Y^I_{t_{m+1}}\Big] + \int_{t_m}^{t_{m+1}} \mathbb{E}_{t_m}\Big[\frac{1}{2}(\theta^S)^2 - Z^I_{1,s}\theta^S - \frac{(Z^I_{2,s})^2}{2}\Big]ds \\ &\approx \mathbb{E}_{t_m}\Big[Y^I_{t_{m+1}}\Big] + \Big[\frac{1}{2}(\theta^S)^2 - Z^I_{1,t_m}\theta^S - \frac{(Z^I_{2,t_m})^2}{2}\Big]. \end{aligned} \tag{31}$$

As before, the approximation in (31) is rectangular method. By setting $Z^I_{1,t_N} = 0$, $Z^I_{2,t_N} = 0$, $Y^I_{t_N} = (\sum_{i\in I} H^i + nH^D)/\gamma_I$, and using the approximation formulas (29), (30), and (31) together in backward, we can obtain the sample paths of Z^I_{1,t_m}, Z^I_{2,t_m}, and $Y^I_{t_m}$ for $m = N, N-1, \dots, 1, 0$. It is obvious from (26) that

$$\theta^{*R}_{t_m} = Z^I_{2,t_m}. \tag{32}$$

Thus, now we have the sample paths of the equilibrium MPR of systemic mortality risk. The conditional expectations appear in the approximation formulas (29), (30), and (31) are computed by using projection on polynomial basis.

By using the samples paths of θ^{*R} in (32), the sample paths of $\zeta^{\theta^*}_{t_m}$ can also be obtained as follows by approximating the integrals:

$$\zeta^{\theta^*}_{t_m} \approx \exp\Big[-\frac{1}{2}\Big\{(\theta^S)^2 t_m + \Delta t\sum_{k=0}^{m-1}(\theta^R_{t_k})^2\Big\} - \sum_{k=0}^{m-1}(\theta^S\Delta W^S_{t_k} + \theta^{*R}_{t_k}\Delta W^R_{t_k})\Big], \tag{33}$$

for $m = 1, 2, \dots, N$. Then, the price of the longevity bond $B^{\theta^*}_t$ in (4) can be computed as the average of random samples of $\frac{1}{\zeta^{\theta^*}_{t_m}}\zeta^{\theta^*}_{t_N} H^D$.

5.2 Example with two agents

From now on, we focus on the case with two agents $I = \{i_1, i_2\}$ so that $|I| = 2$. Moreover, it is assumed that

$$H^{i_1} = -\int_0^T p(0,t)dt, \qquad H^{i_2} = 0, \tag{34}$$

where $p(t,T)$ is given by (2). Then, (34) means that agent i_1 holds an annuity portfolio, whereas agent i_2's net cash flow without trading is zero.

We introduce a longevity bond whose payoff is given by

$$H^D = \begin{cases} 0 & \text{if } q(0,T) > K_2, \\ \frac{K_2 - q(0,T)}{K_2 - K_1} & \text{if } K_1 \leq q(0,T) \leq K_2, \\ 1 & \text{if } q(0,T) < K_1, \end{cases} \tag{35}$$

where

$$q(0,T) = 1 - p(0,T) = 1 - \exp(-\int_0^T R_t dt).$$

Note that, if $K_1 = 0$ and $K_2 = 1$, then H^D becomes $H^D = 1 - q(0,T) = p(0,T)$ because $0 \leq q(0,T) \leq 1$.

5.3 Numerical results

In our numerical explorations we will investigate two types of longevity bonds. The first bond we will call Zero Coupon Survivorship bond or just Survivorship bond, and this bond we will be characterized by the choice of $K_1 = 0$ and $K_2 = 1$ in the payoff function of Equation 35. The second bond will be a bond that we will call Swiss Re bond motivated by thew works of (cite [FILL]) and, in our case it will be characterized by the choice of $K_1 = 0.5 * q(0,T)$ and $K_2 = 0.7 * q(0,T)$. To calculate bond prices we use $N = 100000$ simulations. Our unreported experiments confirm that for maturity of $T = 1$ and time interval of $1/365$ i.e. of daily length, the price calculations are stable and there is no significant difference compared to choice of $N = 1000000$ and the same time interval.

The Table 1 gives Survivorship and Swiss Re bond prices for maturities of $T = 1$. Prices are calculated for various combinations of b_1 and b_2. An immediate observations is that price of Swiss Re bond for the fixed choice of γ_1, γ_2, b_1 and b_2 parameters is lower than the price of Survivorship bond. This result, given the defined payoffs of two bonds and the fact that Swiss Re bond has bigger set of survivorship outcomes that results in zero payoff, matches an intuitive expectation. For the choice of $b_1 = 0$ i.e. mortality intensity devoid of effects of financial risk, we observe that increase in volatility of mortality of intensity due to longevity risk increases Survivorship bond price. As a general observation we see that increase of financial risk by increasing choice of b_1 parameters, increases both Survivorship and Swiss

				Bond Prices	
γ_a	γ_b	b_1	b_2	Survivorship	Swiss Re
1	1	0.10000	0.10000	0.916256	0.392016
1	1	0.10000	0.01000	0.920422	0.385585
1	1	0.10000	0.00100	0.920444	0.385342
1	1	0.10000	0.00001	0.920442	0.385324
1	1	0.10000	0.00000	0.920442	0.385324
1	1	0.01000	0.10000	0.900742	0.210948
1	1	0.01000	0.01000	0.900630	0.000027
1	1	0.01000	0.00100	0.900594	0.000000
1	1	0.01000	0.00001	0.900591	0.000000
1	1	0.01000	0.00000	0.900591	0.000000
1	1	0.00100	0.10000	0.898796	0.197359
1	1	0.00100	0.01000	0.898559	0.000000
1	1	0.00100	0.00100	0.898523	0.000000
1	1	0.00100	0.00001	0.898520	0.000000
1	1	0.00100	0.00000	0.898520	0.000000
1	1	0.00001	0.10000	0.898581	0.196058
1	1	0.00001	0.01000	0.898333	0.000000
1	1	0.00001	0.00100	0.898297	0.000000
1	1	0.00001	0.00001	0.898294	0.000000
1	1	0.00001	0.00000	0.898294	0.000000
1	1	0.00000	0.10000	0.898579	0.196045
1	1	0.00000	0.01000	0.898331	0.000000
1	1	0.00000	0.00100	0.898295	0.000000
1	1	0.00000	0.00001	0.898292	0.000000
1	1	0.00000	0.00000	0.898292	0.000000

Table 1: Survivorship and Swiss Re bond prices for maturity of $T = 1$ given $\gamma_a = 1$ and $\gamma_b = 1$ for various choices of b_1 and b_2.

Re bond prices. Also, for a fixed value of financial risk (i.e fixed value of parameter b_1) and increase in longevity risk (increase in value of parameter b_2), increases the Survivorship bond price and vice versa.

In the Table 2 the volatility of mortality intensity, by setting $b_1 = 0$, is not affect by the financial risk but depends exclusively on longevity risk. The Survivorship and Swiss Re bond prices for maturities of $T = 1$ are given. We observe that for any fixed combination of γ_a and γ_b, i.e. levels of risk aversion, Survivorship bond prices increase as volatility coefficient b_2 increases. For any fixed choice of γ_a and γ_b the inverted choice of γ_b and γ_a has the same price. For example price of Survivorship bond for $\gamma_a = 1$ and $\gamma_b = 10$ is the same as Survivorship price for $\gamma_a = 10$ and $\gamma_b = 1$. In addition, for a fixed choice of $b_2 = 0.1$ the Swiss

Re bond price is non decreasing function of the aggregate risk aversion i.e. $\gamma_a + \gamma_b$. Here, all observation coincide with the common intuition.

6 Concluding Remark

We consider a model with stochastic mortality intensity that affects the income of agents in the market, and introduce a longevity bond market which can be used to hedge the idiosyncratic risk of stochastic mortality intensity. By using Clark-Haussmann formula, we prove that our longevity bond market completes the market. We also provide numerical results for two type of longevity bonds.

				Bond Prices	
γ_a	γ_b	b_1	b_2	Survivorship	Swiss Re
1	1	0	0.1	0.898579	0.196045
1	1	0	0.01	0.898331	0.000000
1	1	0	0.001	0.898295	0.000000
1	1	0	1.00E-05	0.898292	0.000000
1	1	0	0	0.898292	0.000000
1	10	0	0.1	0.898603	0.224382
1	10	0	0.01	0.897798	0.000000
1	10	0	0.001	0.897752	0.000000
1	10	0	1.00E-05	0.897748	0.000000
1	10	0	0	0.897748	0.000000
1	100	0	0.1	0.898608	0.230470
1	100	0	0.01	0.897693	0.000000
1	100	0	0.001	0.897645	0.000000
1	100	0	1.00E-05	0.897641	0.000000
1	100	0	0	0.897641	0.000000
10	1	0	0.1	0.898603	0.224382
10	1	0	0.01	0.897798	0.000000
10	1	0	0.001	0.897752	0.000000
10	1	0	1.00E-05	0.897748	0.000000
10	1	0	0	0.897748	0.000000
10	10	0	0.1	0.898606	0.227436
10	10	0	0.01	0.897745	0.000000
10	10	0	0.001	0.897698	0.000000
10	10	0	1.00E-05	0.897694	0.000000
10	10	0	0	0.897694	0.000000
10	100	0	0.1	0.898608	0.230532
10	100	0	0.01	0.897692	0.000000
10	100	0	0.001	0.897644	0.000000
10	100	0	1.00E-05	0.897640	0.000000
10	100	0	0	0.897640	0.000000
100	1	0	0.1	0.898608	0.230470
100	1	0	0.01	0.897693	0.000000
100	1	0	0.001	0.897645	0.000000
100	1	0	1.00E-05	0.897641	0.000000
100	1	0	0	0.897641	0.000000
100	10	0	0.1	0.898608	0.230532
100	10	0	0.01	0.897692	0.000000
100	10	0	0.001	0.897644	0.000000
100	10	0	1.00E-05	0.897640	0.000000
100	10	0	0	0.897640	0.000000
100	100	0	0.1	0.898609	0.230844
100	100	0	0.01	0.897686	0.000000
100	100	0	0.001	0.897638	0.000000
100	100	0	1.00E-05	0.897634	0.000000
100	100	0	0	0.897634	0.000000

Table 2: Survivorship and Swiss Re bond prices for maturity of $T = 1$ given various choices of γ_a, γ_b, b_1 and b_2.

A Proof of equivalence

Proof. Since $\bar{\pi}^i = (\bar{\pi}^i_1, \bar{\pi}^i_2)$ is the optimal strategy for the optimization problem (9), we have

$$\bar{Y}^i_t(\pi^i) \geq \bar{Y}^i_t(\bar{\pi}^i) \tag{36}$$

For any admissible strategy $\check{\pi}^i$ for the new model, we have

$$\begin{aligned}
&\check{Y}^i_t(\check{\pi}^i) + \nu^i B^\theta_t \\
&\quad = \check{Y}^i_t(\check{\pi}^i) + \nu^i\Big(H^D - \int_t^T \kappa^{\theta,S}_s(dW^S_s + \theta^S_s ds) - \int_t^T \kappa^{\theta,R}_s(dW^R_s + \theta^R_s ds)\Big) \quad \text{(by Eq (5))} \\
&\quad = -H^i + \int_t^T \Big(g^i(s, Z^a_s) - \check{\pi}^i_{1,s}\sigma^S_s\theta^S_s S_s - (\check{\pi}^i_{2,s} + \nu^i)(\kappa^{\theta,}_s\theta^S_s + \kappa^{\theta,R}_s\theta^R_s)\Big)\, ds \\
&\qquad - \int_t^T \Big(Z^i_{1,s} + \check{\pi}^i_{1,s}\sigma^S_s S_s + (\check{\pi}^i_{2,s} + \nu^i)\kappa^{\theta,S}_s\Big)\, dW^S_s \\
&\qquad - \int_t^T \Big(Z^i_{2,s} + (\check{\pi}^i_{2,s} + \nu^i)\kappa^{\theta,R}_s\Big)\, dW^R_s \qquad \text{(by the definition of } \check{Y}^a_t\text{)} \\
&\quad = \bar{Y}^i_t((\check{\pi}^i_{1,s}, \check{\pi}^i_{2,s} + \nu^i)) \qquad \text{(by the definition of } \bar{Y}^i_t\text{)} \\
&\quad \geq \bar{Y}^i_t((\bar{\pi}^i_1, \bar{\pi}^i_2)), \qquad \text{(by the optimality of } \bar{\pi}^i \text{ in (36))} \\
&\quad = -H^i + \int_t^T \Big(g^i(s, Z^a_s) - \bar{\pi}^i_1\sigma^S_s\theta^S_s S_s - \bar{\pi}^i_2(\kappa^{\theta,}_s\theta^S_s + \kappa^{\theta,R}_s\theta^R_s)\Big)\, ds \\
&\qquad - \int_t^T \Big(Z^i_{1,s} + \bar{\pi}^i_1\sigma^S_s S_s + \bar{\pi}^i_2\kappa^{\theta,S}_s\Big)\, dW^S_s - \int_t^T \Big(Z^i_{2,s} + \bar{\pi}^i_2\kappa^{\theta,R}_s\Big)\, dW^R_s \quad \text{(by the definition of } \bar{Y}^i_t\text{)} \\
&\quad = -(H^i + \nu^i H^D) + \int_t^T \Big(g^i(s, Z^i_s) - \bar{\pi}^i_1\sigma^S_s\theta^S_s S_s - (\bar{\pi}^i_2 - \nu^i)(\kappa^{\theta,}_s\theta^S_s + \kappa^{\theta,R}_s\theta^R_s)\Big)\, ds \\
&\qquad - \int_t^T \Big(Z^i_{1,s} + \bar{\pi}^i_1\sigma^S_s S_s + (\bar{\pi}^i_2 - \nu^i)\kappa^{\theta,S}_s\Big)\, dW^S_s - \int_t^T \Big(Z^i_{2,s} + (\bar{\pi}^i_2 - \nu^i)\kappa^{\theta,R}_s\Big)\, dW^R_s \\
&\qquad + \nu^i\Big(H^D - \int_t^T \kappa^{\theta,S}_s(dW^S_s + \theta^S_s ds) - \int_t^T \kappa^{\theta,R}_s(dW^R_s + \theta^R_s ds)\Big) \\
&\quad = \check{Y}^i_t((\bar{\pi}^i_1, \bar{\pi}^i_2 - \nu^i)) + \nu^i B^\theta_t. \quad \text{(by the definition of } \check{Y}^i_t \text{ and Eq (5))}
\end{aligned}$$

Above inequality shows that

$$\check{Y}^i_t(\check{\pi}^i) \geq \check{Y}^i_t((\bar{\pi}^i_1, \bar{\pi}^i_2 - \nu^i))$$

for any admissible strategy $\check{\pi}^i$, and this means that, if $(\bar{\pi}^i_1, \bar{\pi}^i_2)$ is optimal for the original model, then $(\bar{\pi}^i_1, \bar{\pi}^i_2 - \nu^i)$ is the optimal strategy for the new model. Therefore, we have the equation (16). By using similar argument, we can also show that, if $(\tilde{\pi}^i_1, \tilde{\pi}^i_2)$ is optimal for the new model, then $(\tilde{\pi}^i_1, \tilde{\pi}^i_2 + \nu^i)$ is optimal for the original model. Hence we have the equation (17). $\square$

B Proof of Theorem 1

Proof. To prove Theorem 1, we only need to show that the longevity bond completes the market because other parts of the theorem are already proved in the main text. To show that the longevity bond completes the market, it is sufficient to establish $\kappa^{\theta^*,R}$ is not zero a.s..

Let us define the state process $X_t = (X_{1,t}, X_{2,t}, X_{3,t})$ as follows

$$X_{1,t} = \ln S_t, \quad X_{2,t} = \int_0^t R_u \, du, \quad X_{3,t} = R_t$$

Recall that it is assumed that θ_t^S is a function of t, S_t, and R_t. Thus it can be considered as a function of t and the state process$X_t = (X_{1,t}, X_{2,t}, X_{3,t})$ as follows

$$\theta_t^S = \Theta^S(t, X_t) = \Theta^S(t, X_{1,t}, X_{2,t}, X_{3,t}).$$

We will proceed with the following steps:

- Show that θ_t^{*R} is a function of t and X_t, i.e.,

$$\theta_t^{*R} = \Theta^R(t, X_t) = \Theta^R(t, X_{1,t}, X_{2,t}, X_{3,t}).$$

- Establish some differentiability of the map $\Theta^R(t, \cdot, \cdot, \cdot)$. We may follow the arguments of Horst et al. (2010).

- Use Clark-Haussmann formula to derive $\kappa^{\theta^*,S}$ and $\kappa^{\theta^*,R}$.

The first step follows from the Markovian structure of our model. As for the second step we establish the following Lemma.

Lemma 2. *The function $x \mapsto \Theta^R(t, x)$ is Lipschitz continuous uniformly on compact time intervals.*

Proof. We can follow line by line the proof given in Horst et al. (2010). Here we present just the main ideas. The first idea is to show that θ_t^{*R} solve a BSDE which is obtained by taking Malliavin derivative of BSDE (24). Then by Theorem 3.8 in Kobylansky (2000), it follows that $\Theta^R(t, x)$ is the viscosity solution of a PDE. Finally, Theorem 3.3 (b) of Jakobsen and Karlsen (2002) yields the conclusion. $\square$

Let us move to the third step. Recall that $X_t = (X_{1,t}, X_{2,t}, X_{3,t}) = (\ln S_t, \int_0^t R_u \, du, R_t)$, thus

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t^{\theta^*},$$

where

$$f(t, x_1, x_2, x_3) = \begin{pmatrix} -\frac{1}{2} \\ x_2 \\ A(t, x_1, x_2, x_3) \end{pmatrix}, \quad g(t, x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \tag{37}$$

where

$$A(t, x_1, x_2, x_3) = \mu^R x_3 - b_2 \Theta^R(t, x_1, x_2, x_3) - b_1 F(t, e^{x_1}, x_3).$$

Note that Lemma 2 allows us to employ Clark-Haussmann formula to derive $\kappa^{\theta^*,S}$ and $\kappa^{\theta^*,R}$. Consider $\Phi(s,t)$, the solution of the following first variation equation associated with X_t:

$$d\Phi(t, s) = f_x(t, X_t)\Phi(t, s)dt + g_x^1(t, X_t)\Phi(t, s)dW_t^S + g_x^2(t, X_t)\Phi(t, s)dW_t^{*R}, \ t > s, \quad \Phi(s, s) = I_3, \tag{38}$$

where I_3 is the 3×3 identity matrix,

$$f_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ A_{x_1} & A_{x_2} & A_{x_3} \end{pmatrix}, \tag{39}$$

and g_x^1 and g_x^2 are 3×3 matrices with zero entries (g^j means the j-th column of the matrix g in (37)). Thus, (38) is indeed

$$d\Phi(t, s) = f_x(t, X_t)\Phi(t, s)dt, \ t > s, \quad \Phi(s, s) = I_3.$$

Since

$$d\Phi_{1,1}(t, s) = 0, \quad \Phi_{1,1}(s, s) = 1,$$

$$d\Phi_{1,2}(t, s) = 0, \quad \Phi_{1,2}(s, s) = 0,$$

$$d\Phi_{1,3}(t, s) = 0, \quad \Phi_{1,3}(s, s) = 0,$$

it follows that $\Phi_{1,1}(t, s) = 1$, $\Phi_{1,2}(t, s) = 0$, and $\Phi_{1,3}(t, s) = 0$ for $t > s$. Moreover, since

$$d\Phi_{2,1}(t, s) = \Phi_{2,1}(t, s)dt, \quad \Phi_{2,1}(s, s) = 0,$$

$$d\Phi_{2,3}(t, s) = \Phi_{2,3}(t, s)dt, \quad \Phi_{2,3}(s, s) = 0,$$

we have $\Phi_{2,1}(t,s) = 0$, $\Phi_{2,3}(t,s) = 0$ for $t > s$. We also have $\Phi_{2,1}(t,s) = 0$ for $t > s$ because

$$d\Phi_{3,1}(t,s) = A_{x_3}\Phi_{3,1}(t,s)dt, \quad \Phi_{3,1}(s,s) = 0.$$

However, since

$$d\Phi_{2,2}(t,s) = \Phi_{2,2}(t,s)dt, \quad \Phi_{2,2}(s,s) = 1, \tag{40}$$

it follows that $\Phi_{2,2}(t,s)$ is strictly positive for $t \geq s$. Note that, since $\Phi_{1,3}(t,s) = \Phi_{2,3}(t,s) = 0$, we have

$$d\Phi_{3,3}(t,s) = A_{x_3}\Phi_{3,3}(t,s)dt, \quad \Phi_{3,3}(s,s) = 1,$$

and thus, $\Phi_{3,3}(t,s)$ is also strictly positive for $t \geq s$.

Recall that $H^D = G(p(0,T))$, for some monotone increasing (resp. decreasing) function G which is strictly increasing (resp. decreasing) on some interval. By using Clark-Haussmann formula, we obtain $\kappa^{\theta^*,R}$, as follows

$$\kappa_t^{\theta^*,R} = -b_2\mathbb{E}^{\theta^*}\left[G'\left(e^{-\int_0^T R_u du}\right)e^{-\int_0^T R_u du}\int_t^T \Phi_{3,3}(s,t)ds|\mathcal{F}_t\right]. \tag{41}$$

Therefore $\kappa_t^{\theta^*,R}$ is negative (resp. positive). $\square$

References

A. Ang and A. Maddaloni. Do demographic changes affect risk premiums? evidence from international data. Technical report, National Bureau of Economic Research, 2003.

D. Bauer, M. Börger, and J. Ruß. On the pricing of longevity-linked securities. *Insurance: Mathematics and Economics*, 46(1):139–149, 2010.

E. Biffis. Affine processes for dynamic mortality and actuarial valuations. *Insurance: mathematics and economics*, 37(3):443–468, 2005.

D. Blake, A. J. Cairns, and K. Dowd. Living with mortality: Longevity bonds and other mortality-linked securities. *British Actuarial Journal*, 12(01):153–197, 2006.

J. F. Cocco and F. J. Gomes. Longevity risk, retirement savings, and financial innovation. *Journal of Financial Economics*, 103(3):507–529, 2012.

M. M. Dacorogna and M. Cadena. Exploring the dependence between mortality and market risks. *Available at SSRN 2730520*, 2015.

M. Dahl. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. *Insurance: mathematics and economics*, 35(1):113–136, 2004.

G. Deelstra, M. Grasselli, and C. Van Weverberg. The role of the dependence between mortality and interest rates when pricing guaranteed annuity options. *Available at SSRN*, 2015.

C. A. Favero, A. E. Gozluklu, and A. Tamoni. Demographic trends, the dividend-price ratio, and the predictability of long-run stock market returns. *Journal of Financial and Quantitative Analysis*, 46(05):1493–1520, 2011.

U. Horst and M. Müller. On the spanning property of risk bonds priced by equilibrium. *Mathematics of Operations Research*, 32(4):784–807, 2007.

U. Horst, T. A. Pirvu, and G. Dos Reis. On securitization, market completion and equilibrium risk transfer. *Mathematics and Financial Economics*, 2(4):211–252, 2010.

E. Jakobsen and K. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations. *Journal of Differential Equations*, 183:497–525, 2002.

L. Jalen and R. Mamon. Valuation of contingent claims with mortality and interest rate risks. *Mathematical and Computer Modelling*, 49(9):1893–1904, 2009.

M. Kobylansky. Backward stochastic differential equations and partial differential equations with quadratic growth. *Annals of Probability*, 28:558–602, 2000.

D. Lando. *Credit risk modeling: theory and applications.* Princeton University Press, 2009.

E. Luciano and E. Vigna. Mortality risk via affine stochastic intensities: calibration and empirical relevance. *Belgian Actuarial Bulletin*, 8(1):5–16, 2008.

M. Maurer. Asset pricing implications of demographic change. *PhD Thesis, LSE*, 2011a.

T. A. Maurer. Asset pricing implications of demographic change. In *24th Australasian Finance and Banking Conference*, 2011b.

M. Milevsky and D. Promislow. Mortality derivatives and the option to annuitise. *Insurance: Mathematics and Economics*, 29(3):299–318, 2001.

L. Qian, W. Wang, R. Wang, and Y. Tang. Valuation of equity-indexed annuity under stochastic mortality and interest rate. *Insurance: Mathematics and Economics*, 47(2): 123–129, 2010.

K. S. Tan, D. P. Blake, and R. D. MacMinn. Longevity risk and capital markets: The 2013-14 update. *Available at SSRN 2592341*, 2015.