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OIS/dual curve discounting

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OIS/dual curve discounting

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Abstract

This technical note presents the swap pricing paradigm termed dual curve discounting, OIS discounting or CSA discounting that emerged around 2007-2008. We first explain how to apply OIS discounting and then show how the approach is strongly backed by a correct use of no-arbitrage arguments. We conclude by presenting recent developments about OIS discounting in academia and industry.

Introduction

Prior to the 2007 crisis, London Inter Bank Offered Rate (LIBOR) swap rates were used both for discounting and projecting swap cash flows. A single yield curve was estimated and calibrated to liquid market products using the LIBOR swap rates. Cash flows were then estimated and discounted using this yield curve. This practice has however been questioned following the 2007 credit crisis. At that time, LIBOR rates increased substantially with respect to treasury rates, as banks became reluctant to lend to each other amid default concerns. The spread between the 3 month US dollar LIBOR and the 3 month treasury rate, which is usually no greater than 50 basis points, peaked at over 450 basis points in October 2008.

Actually LIBOR rates are not risk-free. They are the short-term borrowing rates of AA-rated financial institutions. As such LIBOR swap rates carry the same risk as a series of short-term loans to such institutions. Therefore in a context of credit risk, liquidity risk and increased use of collateral, LIBOR swap rates are clearly not good proxies for the risk-free rate. OIS (Overnight Indexed Swap) rates, which are associated with a negligible credit risk and value adjustment, are considered a better proxy.

A new pricing paradigm emerged from the crisis. It is referred to as *dual curve discounting*, *OIS discounting* or *CSA* (Credit Support Annex) *discounting*. The variety of names given to it show that it can be understood from different perspectives. Basically, it consists of accounting for the risk premium embedded in the LIBOR. From that perspective, if the LIBOR includes a risk premium, the floating leg of a swap cannot be worth par, and the classical valuation of swaps (which relies on this assumption) must then be fundamentally revisited. Another point of view is that OIS discounting is the natural way, under a no-arbitrage condition, of pricing swaps when collateral is taken into consideration; indeed, in the classical approach for swap valuation, the cost of collateral posting is not taken into consideration, creating arbitrage opportunities.

This technical note is organized as follows. The first section is dedicated to classical swap pricing. It is written in an application-oriented way, in order to allow the reader to quickly see how it works. The second section presents how dual curve discounting is applied, without dwelling on how the formulas are derived. The third section derives dual curve discounting using no-arbitrage arguments. The three sections are independent and need not be read in the proposed order. The last section concludes by directing the reader to recent developments.

1 Classical swap pricing

A swap is a financial contract in which two parties agree to exchange future cash flows. The contract is mainly characterized by its starting and ending dates, reference rate, settlement frequency, notional amount and day count convention. In this paper, we mainly focus on the valuation of fixed against floating swap agreements where parties agree to exchange fixed payments against floating payments. By convention, Party A who receives the fixed payments is called the *receiver*, while his counterparty, Party B, is called the *payer*. The fixed payments are determined by a fixed coupon rate, which is called the *swap rate,* and the notional amount of the swap. The floating payments are determined by the *reference rate* and the same notional amount.

1.1 Examples

1.1.1 Fixed against floating swap

Consider a swap of fixed against 3 month LIBOR initiated on December 4, 2012. The notional is 100M USD. Payments are exchanged every 3 months. The ending date of the swap agreement is December 4 2013. The swap rate is 0.858%. The day count convention is Actual/360. On December 4 2012, the 3-month LIBOR rate was 0.3105%. As a consequence, Party A was due from Party B a payment of 136,877.50 on March 4, 2013. $(90/360 \times (0.858\% - 0.3105\%) \times$ 100M).

1.1.2 Overnight Indexed Swaps

Overnight Indexed Swaps (OIS) are particular fixed against floating swaps, generally of short term, where the reference rate is the Fed fund effective rate or its

Date	Rate	Interest	Accumulated notional
(A)	(B)	(C)	(D)
			$100,000,000.00
1	0.30%	$833.33	$100,000,833.33
2	0.28%	$777.78	$100,001,611.12
3	0.27%	$750.01	$100,002,361.13
4	0.28%	$777.80	$100,003,138.93
5	0.28%	$777.80	$100,003,916.73
Floating payment		$3,916.73	
Fixed payment		$4,166.67	

Figure 1: Computation of the floating payments of an OIS swap
A: Date in days from initiation of contract
B: Fed fund effective rate
C: Interest = $B/360 \times D_{-1}$
D: Accumulated notional $= D_{-1} + C$
Floating payment = sum of column C
Fixed payment = $5/360 \times 0.3\% \times 100M$

equivalent in other markets, such as the EONIA for EUR OIS. As an illustration, consider a 5 days OIS, where party A agrees to pay compounded fed fund effective to party B against a fixed rate of 0.3%. The notional is 100M USD, the day count convention is actual/360, and the fixed and floating payments are exchanged at the expiration of the contract. Figure 1 illustrates how these payments are established on Day 5, after the FED fund effective rates for one day maturity on each day of the contract are known (see column B). For the floating side, the notional accumulates at the FED Fund effective rate and interest is compounded, while the daily interest is constant for the fixed side. The sum of the daily interests is then $\$3,916.73$ for the floating side and $\$4,166.67$ for the fixed side, so that the net payment at the end of the contract is $294.74 from party B to party A.

1.1.3 Basis swaps and basis swap spreads

A swap of two floating rates is called a basis swap. For example, assume Party A agrees to pay over one year the 3-month OIS rate plus 30 basis points to party B against the 3-month LIBOR, on some notional amount. In this example, 30 is called the *basis swap spread*. Basis swaps spreads are determined in such a way that basis swaps are worth par. Basis swaps are widely traded and spreads

Tenor	Bid	Ask	Time
1) 1 Year	17.000	19.000	06:30
2) 2 Year	8.000	10.000	06:30
3) 3 Year	8.000	10.000	06:30
4) 4 Year	8.000	10.000	06:30
5) 5 Year	8.000	10.000	06:30
6) 6 Year	8.000	10.000	06:30
7) 7 Year	8.000	10.000	06:30
8) 8 Year	8.000	10.000	06:30
9) 9 Year	8.000	10.000	06:30
10) 10 Year	8.000	10.000	06:30
11) 20 Year	8.000	10.000	06:30
12) 30 Year	8.000	10.000	06:30

Figure 2: Basis swap spread quotes of 3m vs 6m LIBOR by Tradition, 21/03/2013. Source: Author sceenshot from Bloomberg .

for different tenors are quoted on the markets.

Figure 2 is a Bloomberg screenshot showing basis swap spreads of 3-month LIBOR against 6-month LIBOR quoted by the firm Tradition for different tenors on March 21, 2013. For instance, at that date, the firm was offering to pay 3-month LIBOR plus 17 basis points against 6-month LIBOR over one year, and was also willing to pay 6-month LIBOR against 3-month LIBOR plus 19 basis points over one year.

Basis swaps are widely used for risk management. For instance, a bank funding at the 3-month LIBOR but lending at the 6-month LIBOR rate may use such a swap to hedge its basis risk. The basis swap spread of the Fed rate against LIBOR is an indicator of financial stress. As Figure 3 shows, this spread skyrocketed during the 2007-2009 crisis.

1.2 The pricing formula

It is often useful to view a swap as a contract with two legs. The fixed leg is similar to a fixed coupon bond, where the principal is equal to the notional and the coupon rate is the swap rate. The floating leg pays out periodically the reference rate on the notional of the swap. When both legs are in the same currency, the principal is equal at maturity. The value of the swap is the difference between the present values of the fixed and floating legs. In this section, we illustrate the method for pricing a swap under the classical methodology.

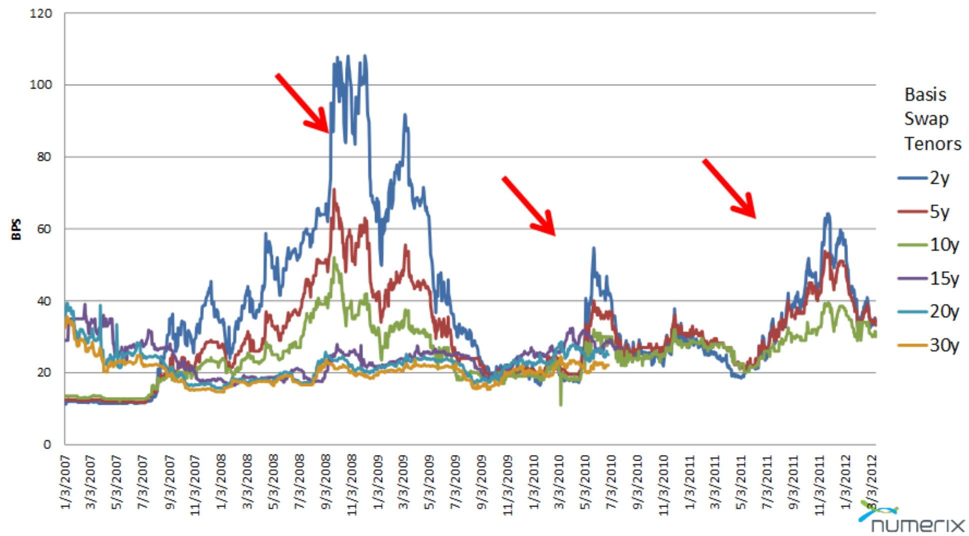

Figure 3: Fed Funds vs LIBOR 3m rate for various swap tenors. Source: Numerix, Satyam Kancharla, May 31 2012.

To simplify the exposition, we normalize the principal to 1 and assume that the fixed and floating payments are exchanged on the same dates, denoted by t_i, $i = 1, ..., n$ (in days), where t_0 is the initiation date of the contract. We use the Actual/360 day count convention and denote by $\Delta_i = \frac{t_i - t_{i-1}}{360}$ the length (in years) of period i of the contract.

Denote the risk-less rate applicable over the period $[t_{i-1}, t_i]$ by r_i.The discount factor used to find the value at date t_0 of cash flows received at date t_i is then given by:

$$\begin{aligned} d_0 &= 1 \\ d_i &= \frac{d_{i-1}}{1 + r_i \Delta_i}, \ i = 1, ..., n. \end{aligned} \tag{1}$$

We define the annuity factor a_i as the sum:

$$\begin{aligned} a_0 &= 0 \\ a_i &= a_{i-1} + d_i \Delta_i. \end{aligned} \tag{2}$$

Now consider a contract with a swap rate c and reference rate L_i during period i. The fixed payment received by Party A at date t_i is $c\Delta_i$, so that its discounted value at t_0 is $c\Delta_i d_i$. Accordingly, the present value of the fixed leg is:

$$\sum_{i=1}^{n} c\Delta_i d_i + d_n = ca_n + d_n.$$

The floating payment received by Party B at date t_i is $L_i\Delta_i$, with discounted value at t_0 equal to $L_i\Delta_i d_i$, so that the present value of the floating leg is:

$$\sum_{i=1}^{n} L_i\Delta_i d_i + d_n. \tag{3}$$

Prior to the 2007 crisis, practitioners considered the LIBOR rate as the riskless investable rate, and therefore discounted cash flows at the LIBOR rate. For a fixed against LIBOR swap, setting $r_i = L_i$ in (1) yields

$$L_i\Delta_i = \frac{d_{i-1}}{d_i} - 1,$$

and the value of the floating leg becomes:

$$\begin{aligned}\sum_{i=1}^{n} L_i\Delta_i d_i + d_n &= \sum_{i=1}^{n}\left(\frac{d_{i-1}}{d_i} - 1\right) d_i + d_n \\ &= \sum_{i=1}^{n}(d_{i-1} - d_i) + d_n \\ &= d_0 - d_n + d_n = 1.\end{aligned}$$

In conclusion, the floating leg of the swap is worth par and, from the position of the fixed receiver, the net present value of the swap is given by the formula:

$$V_0 = ca_n + d_n - 1. \tag{4}$$

Generally the swap rate is determined in such a way that the net present value at initiation is 0. The corresponding rate is the *par swap rate*, which we denote by c^* and is given by:

$$c^* = \frac{1 - d_n}{a_n}. \tag{5}$$

Under this classical approach, the rates L_i used to compute the d_i and a_i at a given evaluation date t_0 are the forward rates applicable over the period $[t_{i-1}, t_i]$ observed at time t_0. It is not surprising that in that case, the floating leg of the swap agreement is worth par, because the discount rate is the same as the rate earned over the payment period. Figure 4 illustrates the computations to price the swap described in section 1.1.1.

1.3 Collateral posting

The above formula (4) can also be used to re-evaluate the swap at any intermediate date t_i by using the forward rates observed at t_i and setting $d_i = 1$ and $a_n = \sum_{j=i}^{n} d_j\Delta_j$. Clearly, because the swap rate is fixed at initiation while the reference forward rates are changing over time, the value of the swap agreement does not stay constant at zero over its life time. If for instance at a given date t_i the value V_i of the swap is positive, the fixed leg is worth more than the

Swap rate	0.858%							
Notional	$100,000,000							
Dates	**Period**	**# of days**	**Forward LIBOR**	**Delta**	**Discount factor**	**Annuity factor**	**Fixed payment**	**Floating payment**
(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	(I)
04/12/2012					1			
04/03/2013	1	90	0.31%	0.25	0.9992	0.2498	$214,503	$77,625
04/06/2013	2	92	0.74%	0.26	0.9973	0.5047	$219,269	$189,347
04/09/2013	3	92	1.04%	0.26	0.9947	0.7589	$219,269	$266,355
04/12/2013	4	91	1.34%	0.25	0.9913	1.0095	$216,886	$337,710
					Present value		$100,000,000	$100,000,000

Figure 4: Evaluation of a swap

A: Payment dates

B: Period

C: number of days in period $=\mathrm{A}-\mathrm{A}_{-1}$

D: 3-month forward LIBOR rate

E: Length of period in years $=\mathrm{C}/360$

F: Discount factor $=\mathrm{F}_{-1}/(1+\mathrm{E}\times\mathrm{D})$

G: Annuity factor $=\mathrm{G}_{-1}+\mathrm{F}\times\mathrm{D}$

H: Fixed payment $=\mathrm{E}\times$ swap rate $\times$ notional

I: Forecasted floating payments $=\mathrm{E}\times\mathrm{D}\times$ notional floating leg at that time. If the fixed payer were to default on his payments, then the fixed receiver would lose V_i. For this reason, provisions in swap agreements require that the fixed payer post the amount V_i as collateral, so that, if he defaults, the receiver can seize the collateral to cover his losses.

Collateral posting is governed by the CSA (Credit Support Annex) that comes along with the swap agreement. Amounts posted as collateral earn interest at the *collateral rate,* which is generally the Fed Fund effective rate, and are adjusted periodically. Accordingly, assuming that the adjustment and payment dates coincide, the net collateral payment of the fixed payer at date t_i is

$$\begin{aligned} C_i &= V_i - V_{i-1}\left(1 + r_i \Delta_i\right), i = 1, ..., n-1 \\ C_n &= -V_{n-1}\left(1 + r_n \Delta_n\right). \end{aligned}$$

2 Dual curve discounting

Dual curve discounting refers to the practice of using one interest rate curve to project the swap cash flows and another curve to discount them. It is also termed OIS discounting because Overnight Indexed Swap rates emerged from the crisis as the correct risk-less investable rate which should be used for discounting. The appellation "CSA discounting" is also often used to reflect the Credit Support Annex that governs collateral posting in swap agreements, because the pricing methodology implied by the new paradigm takes this collateral into consideration, while the classical approach completely ignores it. All three designations refer to the same concept.

2.1 The pricing formula

Consider a basic fixed against floating swap as the one priced by the classical methodology in Section (1.2). The discount rate r_i in the pricing formula of dual curve discounting is the collateral rate, that is, the forward compounded Fed fund effective rates applicable between t_{i-1} and t_i observed at date t_0. These forward rates can be recovered through OIS par swap rates. The formula also takes as input basis swap spreads, as defined in Section (1.1.3), of OIS rate vs. the reference rate. We denote by s_i the basis swap spread, observed at date t_0, of tenor i periods. The process of dual curve discounting consists of constructing the adjusted floating rate from the following equation:

$$\widetilde{L}_i = \frac{s_i a_i - s_{i-1} a_{i-1} - d_i + d_{i-1}}{d_i \Delta_i}, i = 1, ..., n. \tag{6}$$

The floating payments are then projected using the adjusted $\widetilde{L}_i$, and all cash flows are discounted at the discount factors d_i implied by the risk-less forward curve. The net present value of the swap is still computed as the difference between the present value of the fixed and floating fixed legs of the swap:

$$V_0 = c a_n - \sum_{i=1}^{n} \widetilde{L}_i \Delta_i d_i$$

Swap rate	0.923%									
Notional	$100,000,000									
Date	**Period**	**Days**	**Forward FedFund**	**Basis spread**	**Delta**	**Discount factor**	**Annuity factor**	**Floating rate**	**Fixed payment**	**Floating payment**
(A)	**(B)**	**(C)**	**(D)**	**(E)**	**(F)**	**(G)**	**(H)**	**(I)**	**(J)**	**(K)**
04/12/2012						1				
04/03/2013	1	90	0.11%	20	0.25	0.9997	0.2499	0.31%	$230,767	$77,500
04/06/2013	2	92	0.44%	25	0.26	0.9986	0.5051	0.74%	$235,895	$188,847
04/09/2013	3	92	0.60%	32	0.26	0.9971	0.7599	1.06%	$235,895	$270,574
04/12/2013	4	91	0.94%	40	0.25	0.9947	1.0114	1.58%	$233,331	$399,841
							Present value		$100,404,551	$100,404,551

Figure 5: Evaluation of a swap using dual curve discounting
A: Payment dates
B: Period
C: number of days in period $=A-A_{-1}$
D: 3-month forward Fed-Fund rate
E: Basis spread corresponding to the number of periods F: Length of period in years $=C/360$
G: Discount factor $=G_{-1}/(1+F\times D)$
H: Annuity factor $=H_{-1}+F\times G$
I: Adjusted floating rate$=(E\times H\text{-}E_{-1}\times H_{-1}\text{-}G+G_{-1})/(F\times G)$
J: Fixed payment $=F\times$ swap rate $\times$ notional
K: Forecasted floating payments $=F\times I\times$ notional

and the par swap rate is given by[1]:

$$\widetilde{c}^* = \frac{\sum_{i=1}^{n} \widetilde{L}_i \Delta_i d_i}{a_n}. \tag{7}$$

Figure 5 illustrates the computations to price the swap described in section 1.1.1 using dual price discounting.

2.1.1 Explaining the adjusted forward floating rate $\widetilde{L}_i$

Dual curve discounting essentially consists of constructing some adjusted forward reference rate for the projection of the floating cash flows. In this section, we give an interpretation of this adjustment.

First, assume that the basis swap spread s_i is zero for all tenors i. In this case, it is easy to see that the adjusted forward rates are equal to the forward

[1] Notice that this formula uses market-quoted basis swap spreads to compute adjusted libor rates and par swap rates. Formula (7) can also be used to compute adjusted libor rates and basis swap spreads from market-quoted par swap rates.

risk-less rates. Indeed, in this case, equation (6) yields:

$$\begin{aligned}\widetilde{L}_i\Delta_i &= \frac{-d_i + d_{i-1}}{d_i} \\ &= \frac{d_{i-1}}{d_i} - 1 = r_i\Delta_i.\end{aligned}$$

Hence, dual curve discounting reduces to the classical approach when the basis swap spreads between the risk-less and the floating reference rates are zero.

Consider now the case where the basis swap spread is positive. Equation (6) can be rewritten as:

$$\begin{aligned}\widetilde{L}_i\Delta_i &= \frac{-d_i + d_{i-1}}{d_i} + \frac{s_i a_i - s_{i-1}a_{i-1}}{d_i} \\ &= r_i\Delta_i + s_i\frac{a_i}{d_i} - s_{i-1}\frac{a_{i-1}}{d_i},\end{aligned} \quad (8)$$

which shows that the adjusted rate is nothing else than the forward risk-less rate plus a spread. To fix ideas, assume that the floating reference rate is 3-month LIBOR; Basis swap spreads indicate that 3-month LIBOR is exchanged against 3-month OIS rate over $i-1$ periods for s_{i-1} basis points paid at each period, and that 3-month LIBOR is also exchanged against 3-month OIS over i periods for s_i basis points paid at each period. The spread in (8) indicates the value of agreeing at date t_0 to exchange the 3-Month LIBOR against the 3-month OIS for one period between dates t_{i-1} and t_i. It is simply the sum of the s_i basis points paid each period up until period i (properly compounded) minus the sum of the s_{i-1} basis points paid each period up until period i (properly compounded). This can be interpreted as the forward LIBOR-OIS spread.

The forward LIBOR rate L_i of section (1.2) and the adjusted forward LIBOR rate $\widetilde{L}_i$ are both forward LIBOR rates. The first is computed using par fixed vs LIBOR swap rates, while the second is computed using forward OIS rates plus the forward LIBOR-OIS rate spread. The difference between the two forward rates amounts to the discount factors used to compute them from observed data: the construction of the classical forward LIBOR rates L_i uses discount factors implied by LIBOR swap rates, while the construction of the adjusted forward LIBOR uses discount factors implied by par OIS rates.

2.1.2 Some implications of dual curve discounting

As noted earlier, the forward curve used for discounting the cash flows is not that of the floating rate (LIBOR or its equivalent), but rather that of the collateral rate (generally the OIS rate). Moreover, the rates used to project the floating payments are not equal to the "classical" forward floating rate. Now notice that equation (6) defining the adjusted floating rate implies:

$$\sum_{i=1}^{n}\widetilde{L}_i\Delta_i d_i + d_n = 1 + s_n a_n. \quad (9)$$

10

The left hand side of equation (9) is the present value of the floating leg of the swap, where the floating payments are projected at the rate $\widetilde{L}_i$ and discounted with the risk-less discount factor d_i. The right hand side of equation (9) is equal to 1 if the basis swap spread for a tenor of n periods is zero. The intuition is that when the basis spread is zero, the classical approach is valid and the floating leg must be worth par. On the other hand, if the basis swap spread is positive, then the right hand side of equation (9) is greater than one. Therefore, unlike in the classical approach, under dual curve discounting the floating leg of the swap is worth more than par. This difference reflects the risk premium embedded in the floating reference rate, which accounts for the liquidity and default risk of the financial institutions participating in it.

Consider for instance OIS and fixed against LIBOR swaps. OIS bear negligible risk because the notional of swaps is not actually exchanged and because contractual parties have to post collateral. Receiving the LIBOR instead of the OIS rate is a privilege since the former will always be greater, and the floating leg of the LIBOR swap should be worth more than par. Before the 2007 crisis, LIBOR-OIS spreads were close to zero. Even if it was known at that time that the OIS rate was a better proxy for the risk- free rate, there was little need to account for the risk premium because it did not make a significant difference. During the crisis the spread blew up and practitioners were forced to switch from the classical pricing method to dual curve discounting.

In the next section, we show that dual curve discounting is equivalent to no-arbitrage pricing when financing costs are taken into consideration.

3 Dual curve discounting and no-arbitrage pricing

No-arbitrage pricing dictates that two assets generating the same cash flows at the same dates should have the same price. If this were not the case, arbitrageurs would long the cheaper asset and short the other one, and immediately trade away the difference in the two prices. Classical no-arbitrage pricing of a swap consists of constructing a portfolio of traded swaps in such a way that the cash flows of the fixed leg in the replicating portfolio equal those of the fixed leg of the swap. By no-arbitrage arguments, the price of the fixed leg of the swap is then the value of the replicating portfolio.

3.1 Conditions for classical no-arbitrage

Consider a replicating portfolio in the sense that its fixed payments match the fixed payments of the swap. We index by w and by p the variables corresponding to the swap and to the replicating portfolio respectively. Periods are indexed by $i = 1, ..., n$ and dates are denoted by t_i, where t_0 is the initial date. Accordingly, for $j \in \{w, p\}$ and $i = 0, ..., n$, denote by:

M_i^j : notional[2] of instrument j at date t_i,

V_i^j : net present value of instrument j at date t_i,

X_i^j : present value of the fixed leg of instrument j at date t_i,

Δ_i : length of period i in years, $\Delta_i = \frac{t_i - t_{i-1}}{360}$,

f_i^j : floating rate of instrument j observed at date t_{i-1} and applicable during period i,

r_i : collateral rate [3] observed at date date t_{i-1} and applicable during period i.

Consider an arbitrageur who is long the fixed leg of the swap and short the fixed leg of the replicating portfolio. By construction, the fixed payments of the swap and its replicating portfolio offset each other. The cash flow received by the arbitrageur at date t_i is then

$$\begin{aligned} &-\left(f_i^w \Delta_i M_{i-1}^w + (1 + r_i \Delta_i) V_{i-1}^w - V_i^w + M_{i-1}^w - M_i^w\right) \\ &+\left(f_i^p \Delta_i M_{i-1}^p + (1 + r_i \Delta_i) \times V_{i-1}^p - V_i^p + M_{i-1}^p - M_i^p\right). \end{aligned} \qquad (10)$$

From the long side, the arbitrageur makes the floating payment $f_i^w \Delta_i M_{i-1}^w$, returns the previous period collateral plus interest $(1 + r_i \Delta_i) V_{i-1}^w$ and takes the new required collateral V_i^w, and finally makes a fictitious notional payment of $M_{i-1}^w - M_i^w$ so that the balance of the notional is M_i^w. The same logic applies to the short side with the opposite sign.

Under the classical evaluation model, using the definition of the net present value of instrument j at date t_i and the fact that the floating leg is worth par, we have:

$$V_i^j = X_i^j - M_i^j, \; j = w, p. \qquad (11)$$

Moreover, if the law of one price holds, we must have:

$$X_i^w = X_i^p \text{ for all } i. \qquad (12)$$

Substituting equations (11)-(12) into equation (10) yields:

$$M_{i-1}^w (r_i - f_i^w) - M_{i-1}^p (r_i - f_i^p), \qquad (13)$$

which is different from zero in general, unless

$$f_i^w = f_i^p = r_i, \qquad (14)$$

if we exclude the trivial case where the notional of the swap and its replicating portfolio are equal.

Equation (14) says that for the no-arbitrage assumption to be valid, the floating rates of the swap and of its replicating portfolio must be the same, and must equal the collateral rate, otherwise the arbitrageur might have to inject cash flow in order to maintain his positions. If the total cash flow injected exceeded the initial difference in price between the replicating portfolio and the

[2] The notional is allowed to vary through time.

[3] r_i may viewed as the par rate of an OIS initiated at date $t_{i-1}$and ending at date t_i.

fixed leg, he would actually lose money at the closing of his positions. Assuming that the replicating portfolio is constructed using swaps at the same floating rate f_i the total cash flow received by the arbitrageur is

$$(f_i - r_i)\left(M_{i-1}^p - M_{i-1}^w\right).$$

A corollary of this no-arbitrage condition is that classical arbitrage pricing does not work for LIBOR swaps since the LIBOR-OIS spread $f_i - r_i$ is not zero. Another corollary is that classical arbitrage pricing works for OIS swaps because in that case the floating rate and the collateral rate are equal to the Fed fund effective rate or its equivalent.

The intuition for the no-arbitrage condition for is the following. A long position in the fixed leg of a swap has two financing costs: the notional finances at the floating rate, while the net present value of the swap finances at the collateral rate. It follows that there are two sources of financing distortion between the replicating portfolio and the swap. First, the different swaps in the replicating portfolio might not be defined on the same floating rate, and even if it were the case, the floating rate of the replicating portfolio may be different from the floating rate of the swap. Second, assuming that the floating rates are all the same, they are generally different from the collateral rate (except for OIS swaps). These distortions in financing costs show up in the interim cash flows of the arbitrageur.

3.2 No-arbitrage pricing

The reason why fixed vs LIBOR swaps cannot be valued using classical no-arbitrage pricing is that the construction of the replicating portfolio ignores financing costs. Since in practice financing costs are not zero, the replication process should take them into consideration. To use arbitrage pricing for a n-period LIBOR swap, we need to find a way so that the interim cash flows of the arbitrageur given by equation (10) are zero. This can be achieved through a basis swap that exchanges the collateral rate plus a fixed spread against the LIBOR. Receiving a fixed rate c against LIBOR f_i over n periods is equivalent to receiving a fixed rate $c - s_n$ against the collateral rate r_i and receiving the collateral r_i plus the fixed spread s_n against the LIBOR over the same n periods. Thus a long position on the fixed leg of a LIBOR swap is equivalent to a long position on the fixed leg of an OIS swap and a short position on the basis swap. The basis swap spread s_n is determined in such a way that the basis swap is worth par at date 0.

It follows that the value of the n period swap is the sum of the value of the OIS swap and of the basis swap. Since the basis swap is worth par at date 0, the net present value of the n period swap is equal to the net present value of the OIS swap. The discounting is done at the forward collateral rate, which is assumed to equal the forward risk-less rate:

$$V_0 = (c - s_n)\, a_n + d_n - 1, \tag{15}$$

where d_n and a_n are given by (1) and (2).

The no-arbitrage price given by equation (15) is the same as the dual curve discounting price. Indeed equation (15) can be rewritten as

$$V_0 = (ca_n + d_n) - (s_n a_n + 1) \tag{16}$$

The first term in parentheses in the right hand side of equation (16) is the present value of the fixed leg of the swap. The main idea behind dual curve discounting is to compute an adjusted forward floating rate so that the second term in the right hand side of equation (16) can be interpreted as the present value of the floating leg. This can be readily verified by checking the second formulation for the construction of the adjusted forward floating rate given by equation (9).

4 Conclusion

A general presentation of dual curve discounting can be found in Tuckman & Serrat (2012), Douglas and Decrem (2011a,b) and Kancharla (2012). We conclude this brief presentation by reviewing some recent publications on dual curve discounting in both academia and industry.

4.1 Practical issues

This subsection is inspired by an interview of David Kelly, director of the financial engineering department of Calypso, published in Risk magazine.

There are several challenges to the implementation of dual curve discounting in practice. One of them is cross-currency curve construction.

1. There can be an option in the CSA that allows counterparties to post collateral in different currencies. Pricing this option into the OIS curve is a complex issue.

2. Calibration can be also difficult in the case of cross-currency swaps because one has to calibrate an OIS curve in one currency based on the cross currency basis.

3. Some swaps in one currency collateralize in another currency (this is the case of Australian dollar swaps that collateralize in US dollar).

4. Another issue is that of the OIS curve segmentation; one may want for instance to incorporate jumps in the short end of the curve, in order to account for Central Bank meetings.

4.2 Selected recent developments in the industry

Dual curve discounting has created new markets for derivatives valuation and risk management corporations.

SwapClear, the interest rate swap clearing service run by London-based LCH. Clearnet, has begun shifting to OIS discounting in June 2010 and intended in 2011 to extend OIS discounting to its entire multiple currencies swap portfolio (Sawyer 2011).

Principia Partners, a risk management software provider, launched a new version of their platform that is able to support the shift in derivatives markets towards OIS discounting (*Worldwide Computer Products News*, November 9, 2011).

Vancouver-based derivatives valuation and risk management software provider Fincad recently added OIS curves to its platforms. Fincad builds OIS curves for different currencies, including Japanese yen, pounds sterling, euro and Swiss francs, and US dollar (*Inside market Data*, July 9, 2012).

KLP Asset Management, a subsidiary of a Norway-based insurance company that manages NOK 227 billion in assets, replaced its interest rate derivatives vendor by Quantify. The reason of this transaction is their need for a dual curve discounting environment (*Wireless news*, November 6, 2012).

4.3 Selected recent contributions from the academia

A series of working papers have been published on the issue of dual curve discounting in recent years. A first stream in this literature reports on empirical evidence. Bianchetti and Carlicchi (2012) provide extensive evidence of increased use of collateral, divergence between OIS and LIBOR, and explosion of basis swap spreads. They also provide some empirical evidence that the market abandoned the classical approach for the dual curve approach since March 2010. Schwartz (2011) shows that liquidity risk is the predominant factor explaining the LIBOR-OIS spread, ahead of credit risk.

Another group of researchers focus on developing a continuous-time model for the adjusted forward rate. Mercurio (2010) proposes an extension of the one-curve LIBOR Market Model (LMM) with stochastic volatility, where he models the basis between OIS and FRA rates. He then shows that this model is still flexible enough to result in closed-form prices for caps and swaptions. The model is also able to handle simultaneous derivatives of different tenors. Alvarez-Manilla (2012) proposes a non-martingale dynamics for the adjusted forward rate. This contrasts with the works of Bianchetti (2010) and Mercurio (2009) where the adjusted forward rate is a martingale.

No major paper on dual curve discounting has been published in the top-rated financial journals, at least to our knowledge. One may wonder why there is so little interest in academia about dual curve discounting. This lack of interest may be due to the fact that, for researchers, there is little innovation in using OIS rates as risk-less rate, and research is rather focused on the more general problem of credit and liquidity risk. However, several technical issues still remain to be addressed in practice.

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