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CV-Ar-Hedging in Markets with Transaction Costs and its Application

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CVaR-hedging in markets with transaction costs and its applications

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Abstract

This paper analyzes CVaR-hedging and its applications on equity-linked life insurance contracts and on financial regulations in the Jump-Diffusion model with transaction costs. The formula for option value process in markets with transaction costs is proposed. Meanwhile, two kinds of CVaR-based optimal hedging strategies are derived: the one minimizes initial hedging costs subjected to a CVaR hedging constrain and the one that minimizes CVaR with the initial wealth bounded from above. Moreover, we discuss applications of our results by showing how can them be implemented to derive ages of target clients for equity-linked life insurance contracts and to derive the required financial regulatory capital.

JEL classification: C61, G13, G22.

Keywords: Conditional Value-at–Risk, Jump-Diffusion model, Transaction costs, Equity-Linked life insurance contracts, Financial regulations.

1 Introduction

Since the famous paper of Black and Scholes (1973), perfect hedging is a standard and powerful way to price of options. However, when perfect hedging is impossible, a partial hedging strategy that minimizes the shortfall risk should be considered and applied to price of contracts (see a recent book of Melnikov and Nosrati (2017), where these questions are discussed in detail).

The most developed theory of partial hedging deals with financial markets without transaction costs. However, transaction costs are common in real world and in general cannot be ignored. There is a considerable amount of papers devoted to option pricing with transaction costs. Leland (1985) showed that with a modified volatility, one can construct a hedging strategy to replicate the payoff of a European call option. Hodges and Neuberger (1989) designed a utility-based approach to price options with transaction costs. Merton (1990) used a discrete time framework to derive the value of an option with proportional transaction costs. Boyle and Vorst (1992) extended Merton’s analysis to several periods and transformed the Leland approach to a Binomial

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market model. Transaction costs create additional differences, and hedgers cannot hedge continuously, otherwise the transaction costs would be too big. Due to discrete time hedging, dynamic option replication strategies are no longer risk free. Toft (1996) gave closed form solutions for hedging errors and transaction costs for a time-based hedging strategy proposed by Leland (1985). Chow (2017) compared the hedging performance for different hedging strategies with transaction costs. Those papers considered option pricing in the Black- Scholes model.

However, a growing number of evidences show that the diffusion models are not accurate enough and jump components should be taken into consideration. A Jump-Diffusion model for financial needs was proposed by Merton (1976), and now there is a long list of references on this subject (for example, see Amin (1993) developed a discrete time model to value the option in the Jump-Diffusion market model. Mocioalca (2006), Zhou and Han (2015) worked on pricing option in the Jump-Diffusion model with transaction costs).

Our main objective in this paper is to develop the CVaR-hedging approach in a Jump Diffusion model with transaction costs. This paper is organized as follows: in Section 2, the Jump-Diffusion model is introduced, and some of its properties are discussed. Moreover, the value process of a European call option in the Jump-Diffusion market model with transaction costs is derived. In Section 3.1, we start with the definition of conditional value-at-risk and then in Section 3.2 with the help of optimal CVaR-hedging techniques developed by Melnikov and Smirnov (2012) in the Black-Scholes environment, the optimal strategy that minimizes hedging costs subjected to a CVaR constrain for a European call option in the Jump-Diffusion model is derived. Meanwhile, in Section 3.3, the problem of minimizing CVaR with an initial wealth bounded from above is discussed. In Section 4, applications of CVaR on equity-linked life insurance contracts and financial regulations are shown. In Section 4.1, a numerical example is given to illustrate how our method in Section 3.2 can be used to find the target clients' age of an embedded call option contract, whereas in Section 4.2, we show how the method in Section 3.3 can be implemented to calculate the CVaR-based capital requirement in the area of financial regulation. In Section 5, we give the conclusion for the paper.

2 Description of the model and option pricing in markets with transaction costs

2.1 **Model setup**

Let $(\Omega, \mathcal{F}, F = (\mathcal{F}_t)_{t\geq 0}, P)$ be a standard stochastic basis. Considering a financial market with one risk less asset $(B_t)_{t\geq 0}$, and two risky assets, $(S_t^1)_{t\geq 0}$, $(S_t^2)_{t\geq 0}$, such that, they satisfy the following stochastic differential equations:

$$dB_t = rB_t dt, B_0 = 1,$$

$$dS_t^i = S_t^i(\mu_i dt + \sigma_i dW_t - \upsilon_i d\mathrm{N}_t), \quad i = 1,2 \ , \tag{2.1}$$

where $r \geq 0$ is the risk-free interest rate, $\mu_i \in R$, $\sigma_i > 0$, $\upsilon_i < 1$, $(W)_{t_{t\geq 0}}$, and $(\mathrm{N}_t)_{t\geq 0}$ are independent Winer and Poisson processes. Assume the filtration F is generated by W and N.

First, let us consider the market without transaction costs and discuss some of its properties.

The market (2.1) is complete if the following conditions are fulfilled (see, for instance, Melnikov and Nosrati (2017))

$$\frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{\nu_1\sigma_2 - \nu_2\sigma_1} > 0, \qquad \nu_1\sigma_2 - \nu_2\sigma_1 \neq 0.$$

Such a market admits a unique martingale measure P^* with the following local density:

$$Z_t = \frac{dP^*}{dP}\bigg|_{\mathcal{F}_t} = exp\left(\alpha^* W_t - \frac{\alpha^{*2}}{2}t + (\lambda - \lambda^*)t + (ln\lambda^* - ln\lambda)N_t\right), \tag{2.2}$$

where λ is an intensity parameter of N under measure P, and the pair (α^*, λ^*) satisfies:

$$\lambda^* = \frac{(\mu_1 - r)\sigma_2 - (\mu_2 - r)\sigma_1}{\nu_1\sigma_2 - \nu_2\sigma_1}$$

$$\alpha^* = \frac{(\mu_1 - r)v_2 - (\mu_2 - r)v_1}{\nu_1\sigma_2 - \nu_2\sigma_1}$$

Moreover, under the martingale measure P^*, process $W_t^* = W_t - \alpha^* t$ and N_t are again independent Wiener and Poisson process (with intensity λ^*).

We can obtain the exponential representation of S_t^i, $i = 1,2$:

$$S_t^i = S_0^i exp(\sigma_i W_t + [\mu_i - \frac{1}{2}\sigma_i{}^2]t + \mathrm{N}_t \ln(1 - \nu_i))$$

$$= S_0^i \, exp\left(\sigma_i W_t^* + \left[\nu_i \lambda^* - \frac{1}{2}{\sigma_i}^2\right] t + \mathrm{N}_t \ln(1-\nu_i)\right). \quad (2.3)$$

We assume that a hedger is exposed to a future obligation $H = (S_T^1 - K)^+$ at maturity time T. At any time $t \le T$, denote the value of a portfolio π as V_t^π. Define a shortfall risk $L = H - V_T^\pi$, and formulate following problems under consideration in terms of L.

2.2 Option pricing in markets with transaction costs

Assuming buying and selling stocks needs to pay transaction costs which are proportional to the volume of transactions. i.e.: $k|w|S$, where $|w|$ represents the share of the trading underlying risky asset, and k represents a fixed proportion of the transaction fee. We assume that the replicated portfolio is rebalanced at discrete intervals and trades can only be executed at certain points of time $\{t_0, t_1 \dots t_M\}$, $t_M = T$. The time interval Δt between two transactions is fixed. Let us denote the value process of a claim as $\bar{V}(t) = \bar{V}(t, S_t^1, S_t^2)$.

Lemma 2.1 In markets with transaction costs, the value process $\bar{V}(t)$ of an option with underlying assets following model (2.1) satisfies the partial differential equation:

$$\bar{V}_t + \frac{1}{2}\bar{V}_{s^1 s^1} {s_t^1}^2 \hat{\sigma}_1 + \frac{1}{2}\bar{V}_{s^2 s^2} {s_t^2}^2 \hat{\sigma}_2 + \frac{1}{2}\sigma_1\sigma_2 \bar{V}_{s^1 s^2} s_t^1 s_t^2 + (r + \lambda v_1) s_t^1 \bar{V}_{s^1}$$

$$+ (r + \lambda v_2) s_t^2 \bar{V}_{s^2} - r\bar{V}(t) + \lambda(\bar{V}_+(t) - \bar{V}(t)) = 0, \quad (2.4)$$

where

$$\bar{V}_t = \frac{\partial \bar{V}(t)}{\partial t},\ \bar{V}_{s^1} = \frac{\partial \bar{V}(t)}{\partial S_t^2},\ \bar{V}_{s^2} = \frac{\partial \bar{V}(t)}{\partial S_t^2},\ \bar{V}_{s^1 s^2} = \frac{\partial^2 \bar{V}(t)}{\partial S_t^1 \partial S_t^2},$$

$$\bar{V}_+(t) = \bar{V}_+(t, S_t^1, S_t^2) = \bar{V}(t, (1-v_1)S_t^1, (1-v_2)S_t^2),$$

$${\hat{\sigma}_i}^2 = {\sigma_i}^2\left(1 + \frac{2k}{\sigma_i}\sqrt{\frac{2}{\pi \Delta t}}\, sign(\bar{V}_{s^i s^i}) - \frac{2}{{\sigma_1}^2} k \lambda v_i\, sign(\bar{V}_{s^i s^i})\right), (i = 1,2),$$

$$sign(x) = \begin{cases} 1, & if\ \ x \ge 0, \\ -1, & if\ \ x < 0. \end{cases}$$

Proof: see Appendix A.

Specifically, by Lemma 2.1, the value process for a call option $H = (S_T^1 - K)^+$ can be derived.

Proposition 2.2 In markets with transaction costs, at each revision point t_m, $m = 0, 1, \dots M-1$, the adjusted value process $\bar{V}(t_m)$ of the contingent claim $H = (S_T^1 - K)^+$ satisfies:

$$\bar{V}(t_m)= V\left(t_m, S_{t_m}^1, \hat{\sigma}_1, \lambda\right)=\sum_{n=0}^{\infty} \bar{C}^{\mathrm{BS}}\left(\mathrm{S}_{\mathrm{t}}^1 \upsilon_{\mathrm{n,T_m}}, K, \mathrm{t_m}\right) p_{n,\mathrm{T_m}}, \quad (2.5)$$

where $V(t, S_t^1, \sigma_1, \lambda^*)$ is the value process for the same claim in markets without transaction costs. λ is the intensity for possion process under measure P,

$$\mathrm{T_m} = T - t_m,$$

$$\bar{C}^{\mathrm{BS}}(S_0, K, t) = S_0 \Phi\left(\bar{d}_+(S_0, K, t)\right) - e^{-rT} K \Phi\left(\bar{d}_-(S_0, K, t)\right),$$

where $\Phi(\mathrm{x})$ is the standard normal distribution,

$$\bar{d}_{\pm}(S_0, K, t) = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r \pm \frac{\hat{\sigma}_1}{2}\right)(\mathrm{T-t})}{\hat{\sigma}_1 \sqrt{\mathrm{T-t}}},$$

$$p_{n,\mathrm{t}} = \mathrm{e}^{-\lambda \mathrm{t}} \frac{(\lambda \mathrm{t})^n}{n!},$$

$$\upsilon_{\mathrm{n,t}} = (1 - v_1)^n \, exp(v_1 \lambda \mathrm{t}).$$

Proof: See Appendix A.

3 Optimal CVaR-hedging

3.1 Conditional Value-at-Risk (CVaR)

Definition 3.1 The CVaR of risk L at level α is defined as:

$$CVaR_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^\infty VaR_s(L) \, ds \quad (3.1)$$

where $VaR_\alpha = inf\{t \in R : P(L \leq t) > \alpha\}$.

$CVaR_\alpha(L)$ represents the expected loss for a hedging strategy given that the loss exceeds its upper α quantile.

In our paper, we consider a European call option based on the first stock $H = (S_T^1 - K)^+$. We would like to construct optimal hedging strategies based on conditional value-at-risk (CVaR). Two dual versions of the problem are considered: minimization of hedging costs subject to a CVaR constrain and minimization of CVaR subject to an initial value constrain.

3.2 Minimizing initial costs

In this section we provide an approach to solve the problem of minimizing hedging costs with a CVaR constrain in the Jump-Diffusion market model (2.1). i.e.:

$$\begin{cases} min \ V_{0,} \\ CVaR_\alpha(L) \leq \bar{C}, \end{cases} \quad (3.2)$$

where $L = H - V_T^\pi$ is the hedging risk, V_0 is the initial wealth, and $\bar{C}$ is a fixed CVaR constrain.

In Melnikov and Smirnov (2012), the solution for such a problem in markets without transaction costs was given by using the Neyman-Pearson Lemma. For readers convenience, the main result of their paper is summarized as follows:

The optimal strategy $(\hat{V}_0, \hat{\xi})$ for the problem (3.2) is a perfect hedging strategy for the contingent claim $(H - \hat{z})^+\big(1 - \hat{\varphi}(\hat{z})\big)$, if condition:

$$E_P(H) > \bar{C}(1-\alpha), \quad E_P((H-\bar{C})^+) > 0, \tag{3.3}$$

holds true, where $\hat{\varphi}(z)$ is defined by:

$$\tilde{\varphi}(z) = 1_{\left\{\frac{dP^*}{dP} > \tilde{a}(z)\right\}} + \gamma(z) 1_{\left\{\frac{dP^*}{dP} = \tilde{a}(z)\right\}}, \tag{3.4}$$

$$\tilde{a}(z) = inf\left\{a \geq 0\text{:}\ E_P\left[(H-z)^+ 1_{\left\{\frac{dP^*}{dP} > a\right\}}\right] \leq (\bar{C} - z)(1-\alpha)\right\}, \tag{3.5}$$

$$\gamma(z) = \frac{(\bar{C}-z)(1-\alpha) - \ E_P\left[(H-z)^+ 1_{\left\{\frac{dP^*}{dP}\tilde{a}(z)\right\}}\right]}{E_P\left[(H-z)^+ 1_{\left\{\frac{dP^*}{dP} = \tilde{a}(z)\right\}}\right]}\ . \tag{3.6}$$

$\hat{z}$ is a point of minimum of function:

$$d(z) = E_{P^*}\left[(H-z)^+\big(1 - \tilde{\varphi}(z)\big)\right],$$

on the interval $z \in (0,\ \bar{C})$.

3.2.1 Optimal hedging strategy and its value process in markets with transaction costs

By relations (3.4)-(3.6), the optimal hedging strategy with CVaR constraints for the claim $H = (S_T^1 - K)^+$ can be derived.

Lemma 3.2 Under CVaR restriction $CVaR_\alpha(L) \leq \bar{C}$, the optimal CVaR-hedging strategy for $H = (S_T^1 - K)^+$ in markets without transaction costs is a perfect hedge for the claim $(H - \hat{z})^+ 1_{\left\{S_T^1 > \hat{m} b^{*N_t}\right\}}$, where $\hat{z},\ \hat{m}$ are defined as following:

$m(z)$ is the unique solution for the system:

$$\begin{cases} n_a(m,z) = inf\{n, mb^{*n} \geq K(z)\}, \\ \displaystyle\sum_{n=n_a(m,z)}^{\infty} (\tilde{S}_{0,n}^1 e^{\mu_1 T} \wedge_+ (K(z), mb^{*n}) - K(z) \wedge_- (K(z), mb^{*n}))p_{n,T} \\ = (\bar{C} - z)(1-\alpha), \end{cases} \tag{3.7}$$

$$b^* = \left(\frac{\lambda^*}{\lambda}\right)^{-\frac{\sigma_1}{\alpha^*}} (1 - v_1),$$

$$K(z) = k + z,$$

$$\tilde{S}_{0,n}^{1} = S_0^1(1-v_1)^n,$$

$$\Lambda_{\pm}\ (K(z), mb^{*n}) = \Phi\left(d_{\pm}^{1}\left((\tilde{S}_{0,n}^{1}, K(z)\right)\right)\text{-}\ \Phi\left(d_{\pm}^{1}\left((\tilde{S}_{0,n}^{1}, mb^{*n}\right)\right),$$

$$d_{\pm}^{1}(S_0, K) = \frac{\ln\left(\frac{S_0}{K}\right)+\left(\mu_1 \pm \frac{\sigma_1^2}{2}\right)T}{\sigma_1\sqrt{T}},$$

$\hat{z}$ is the point of minimum for function:

$$d(z) = \sum_{n=0}^{n_a(m(z),z)-1} \mathrm{C}^{\mathrm{BS}}\left(\hat{S}_{0,n}^{1}, K(z), T\right) p_{n,T}^{*}$$

$$+ \sum_{n=n_a(m(z),z)}^{\infty} \mathrm{C}^{\mathrm{BS}}\left(\hat{S}_{0,n}^{1}, m(z)b^{*n}, T\right) p_{n,T}^{*}\ ,$$

where $$\mathrm{C}^{\mathrm{BS}}(S_0, K, T) = S_0\Phi\big(d_+(S_0, K)\big) - e^{-rT}K\Phi\big(d_-(S_0, K)\big),$$

$$d_{\pm}(S_0, K) = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r \pm \frac{\sigma_1^2}{2}\right)T}{\sigma_1\sqrt{T}},$$

$$\hat{S}_{0,n}^{1} = S_0^1(1-v_1)^n\, exp\big((v_1\lambda^*\mathrm{T})\big),$$

$$p_{n,T}^{*} = \mathrm{e}^{-\lambda^*\mathrm{T}}\frac{(\lambda^*\mathrm{T})^n}{n!},$$

and $\widehat{m} = m(\hat{z})$.

Moreover, the CVaR-hedging price for the claim $(H-\hat{z})^{+}1_{\left\{S_T^1 > \widehat{m}b^{*\mathrm{N}t}\right\}}$ is:

$$V_0 = \sum_{n=0}^{n_a(\widehat{m},\hat{z})-1} C^{BS}\left(\hat{S}_{0,n}^{1}, K(\hat{z}), T\right) p_{n,T}^{*}$$

$$+ \sum_{n=n_a(\widehat{m},\hat{z})}^{\infty} [\hat{S}_{0,n}^{1}\Phi\left(d_+\left(\hat{S}_{0,n}^{1}, m(z)b^{*n}\right)\right) - e^{-rT}K(z)\Phi\left(d_-\left(\hat{S}_{0,n}^{1}, m(z)b^{*n}\right)\right)]p_{n,T.}^{*}$$

(3.8)

Proof: See Appendix B.

In our paper, for illustration purposes, we only consider the case $\alpha^* < 0$, and $b^* \geq 1$.

By Proposition 2.2, the value process of CVaR optimal hedging strategy in the market with transaction costs can also be derived.

Lemma 3.3 In the market with transaction costs (model (2.1)), at each revision point t_m, $m = 0,1, \dots M-1$, the adjusted value process $\bar{V}(t_m)$ of the contingent claim $(H-\hat{z})^{+}1_{\{S_T^1 > \hat{m} b^{*N_t}\}}$ satisfies:

$$\bar{V}(t_m) = \sum_{n=0}^{n_p(t)-1} \bar{C}^{BS}\left(S_t^1 \upsilon_{n,T_m}, K(\hat{z}), T_m\right) p_{n,T_m}$$
$$+ \sum_{n_p(t)}^{\infty} [S_t^1 \upsilon_{n,T_m} \Phi\left(\bar{d}_+\left(S_t^1 \upsilon_{n,T_m}, M_{n,t_m}, t_m\right)\right)$$
$$-e^{-rT} K(z) \Phi\left(\bar{d}_-\left(S_t^1 \upsilon_{n,T_m}, M_{n,t_m}, t_m\right)\right), \qquad (3.9)$$

where

$$M_{n,t_m} = \hat{m} b^{*N_{t_m}} (b^*)^n ,$$
$$n_p(t_m) = inf\{n, M_{n,t_m} \geq K(\hat{z})\}.$$

This pricing formula indicates that at each revision point t_m, $m = 1, \dots, M-1$, the shares of risky asset in the hedging portfolio is given by:

$$\varphi_{t_m}^1 = \frac{\partial \bar{V}(t)}{\partial S_t^1} = \sum_{n=0}^{n_p(t)-1} \upsilon_{n,T_m} p_{n,T_m} \Phi\left(\bar{d}_+\left(S_t^1 \upsilon_{n,T_m}, K(\hat{z}), t_m\right)\right)$$
$$+ \sum_{n_p(t)}^{\infty} \upsilon_{n,T_m} p_{n,T_m} (\Phi\left(\bar{d}_+\left(S_t^1 \upsilon_{n,T_m}, M_{n,t_m}, t_m\right)\right)$$
$$+exp(rT_m)\left(M_{n,t_m} - K(\hat{z})\right) n\left(\bar{d}_+\left(S_t^1 \upsilon_{n,T_m}, M_{n,t_m}, t_m\right)\right) \frac{1}{S_t^1 \upsilon_{n,T_m} \hat{\sigma}_1 \sqrt{T_m}}), \qquad (3.10)$$

where $n(x)$ is the standard normal density function.

And the value of risk free asset in the hedging portfolio is:

$$\bar{B}_{t_m} = \bar{V}(t_m) - \varphi_{t_m}^1 S_{t_m}^1. \qquad (3.11)$$

3.2.2 Total hedging error and transaction costs

Due to transaction costs, trading can only occur discretely. The hedging position cannot be self-adjusted. Therefore, at each revision point t_{m+1} $(m = 0,1,2 \dots M-1)$, there would be some hedging errors.

Definition 3.4 The hedging errors at time t_{m+1} $(m = 0,1,2 \dots M-1)$ are defined as the difference between the value of hedging portfolio and the price of option:

$$H_{t_{m+1}} = exp(\Delta t)\bar{B}_{t_m} + \varphi_{t_m}^1 S_{t_{m+1}}^1 - \bar{V}(t_m), \qquad (3.12)$$

where $\bar{B}_{t_m}$ is the amount invested in the risk-free asset during time period $t_m - t_{m+1}$. $\varphi^1_{t_m}$ is the number of shares that a hedger holds during time period $t_m - t_{m+1}$ and $\bar{V}(t_m)$ is the option value at time t_m.

The total hedging errors are defined as the sum of hedging errors at each future rebalancing time discounted by risk-free rate:

$$HE = \sum_{n=0}^{M-1} exp(-\mathrm{r}t_{m+1})\, H_{t_{m+1}}. \qquad (3.13)$$

Also, at each revision point, the hedger needs to pay some transaction fees.

Definition 3.5 The transaction costs at each revision point t_{m+1} $(m = 0,1,2 \ldots M - 1)$ are:

$$TC_{t_{m+1}} = kS^1_{t_{m+1}} \left|\varphi^1_{t_{m+1}} - \varphi^1_{t_m}\right|. \qquad (3.14)$$

The same as total hedging errors, we can define total transaction costs as:

$$TC = \sum_{n=0}^{M-1} exp(-\mathrm{r}t_{m+1})\, TC_{t_{m+1}}. \qquad (3.15)$$

Hedging errors $H_{t_{m+1}}$ represent the money a hedger earn at time t_{m+1}, transaction costs $TC_{t_{m+1}}$ represent the money a hedger need to spend at time t_{m+1}. The difference between them can be explained as the net income for him/her at time t_{m+1}. Similarly, $HE - TC$ can be treated as the net income for him/her during the whole hedging period.

The total expected hedging costs consist two parts. The first part is the initial wealth to set up the hedging portfolio, which is determined by the adjusted option price $\bar{V}(t_0)$. The second part is the difference between total expected hedging errors and total expected transaction costs. (i.e $X_0 = \bar{V}(t_0) - (HE - TC)$ can be treated as the total hedging costs to satisfy a CVaR constrain.)

3.3 Minimizing conditional Value-at-risk

In this section we suggest a method to solve the problem of CVaR minimization with an initial wealth bonded from above in the Jump-Diffusion market model:

$$\begin{cases} CVaR_\alpha(L) \to min, \\ \qquad \tilde{V}_0 < V_0, \end{cases} \qquad (3.16)$$

where V_0 is the minimal initial wealth required to hedge perfectly and $\tilde{V}_0$ is the value that the hedger would like to spend.

In Melnikov and Smirnov (2012), they solved such a problem in complete markets. Unfortunately, their method cannot be applied into our case. In our case, the market is no longer complete because of transaction costs. In Cong, Tan, and Weng (2014), they considered the same minimization problem without assuming the completeness of the market. The main result of their paper can be summarized as follows:

The optimal hedging strategy for CVaR minimization problem (3.16) is a bulled call option on the payout H:

$$\widehat{H} = (H - d^*)^+ - (H - u^*)^+ \tag{3.17}$$

where (d^*, u^*) is a point of minimum of 2-dimensional function:

$$\begin{cases} min_{0 \le d \le v, u \ge d} \ d + CVaR_\alpha[(H-u)^+] + e^{rT}\Pi((H-d)^+ - (H-u)^+), \\ \quad s.t \ \Pi((H-d)^+ - (H-u)^+) \le V_0, \end{cases} \tag{3.18}$$

v is the solution for $v = VaR_\alpha(H)$, $\Pi(X_T)$ is the time-0 market price of the contingent claim with payoff X_T at time T.

Advantages of their method are that they do not specify dynamics of underlying assets and do not ask for a specific form of the risk pricing function $\Pi(X_T)$ as long as it preserves the stop-loss order. In our case, it is reasonable to choose the adjusted pricing function mentioned in Lemma 2.1 as Π. We will give a numerical example of this method in Section 4.2.

4 Applications of CVaR-hedging

4.1 CVaR-hedging of Equity-Linked life insurance contracts

Following actuarial traditions, let a random variable $T(x)$ on an "actuarial" probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ denote the remaining life time of a person of current age x. $(\Omega, \mathcal{F}, P)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ are independent. Consider a pure endowment contract with a fixed guarantee K which will pay an insure $\overline{H} = \max\{S_T^1, K\}$ at time T provided the insured is alive.

Notice that, $\overline{H} = \max\{S_T^1, K\} = K + (S_T^1 - K)^+$. It is sufficient for our purpose to only consider the embedded call option $H = (S_T^1 - K)^+$.

In the market without transaction costs, since the mortality risk is essentially independent from the financial market, the premium for such a contract shall be equal to (see Brennan & Schwartz,1976):

$${}_TU_x = E_{P^*}E_{\tilde{P}}\left(H\big(T(x)\big)\right) = E_{P^*}(H)E_{\tilde{P}}\big(1_{\{T(x)>T\}}\big) = {}_Tp_xV_0, \tag{4.1}$$

where x is the insurer's age, T is the maturity time of the contract, ${}_Tp_x = \bar{P}(T(x) > T)$ represents the survival probability for the next T years of the inured, and V_0 is the fair price for embedded call option $H = (S_T^1 - K)^+$.

However, in the market with transaction costs, the price for option H should be recalculated by adjusted volatility and adjusted possion intensity and should also include the expected hedging errors and transaction costs. We denote this adjusted price of H as $\hat{V}_0$. Then, the premium for this contract should be recalculated as ${}_TU_x = {}_Tp_x\hat{V}_0$. Notice that ${}_TU_x \leq \hat{V}_0$, which means the initial amount the insurance company can get is less than the initial wealth they need to hedge perfectly. Therefore, a partial hedging strategy should be constructed. Applying partial hedging into life insurance contract was proposed by Melnikov (2004), and in Melnikov (2005) the application of partial hedging in Jump-Diffusion models was discussed. Here, we assume insurance company would like to implement a CVaR-based partial hedging strategy to satisfy a CVaR constrain. From the work in Section 3.2, the total costs X_0 that required to hedge the claim H under a CVaR constrain can be derived. Therefore, in the market with transaction costs, the initial premium for the claim H should be equal to the total costs that required to construct a CVaR partial hedging strategy, i.e., ${}_TU_x=X_0$.

The important relationship:

$$ {}_TU_x = {}_Tp_x\hat{V}_0 = X_0, \tag{4.2} $$

provides us a way to find the target survival probability ${}_Tp_x=X_0/\hat{V}_0$ (and hence the target clients' age) for the life-insurance contract subjected to a fixed CVaR threshold. We now illustrate this method by a numerical example.

Let the financial part following model (2.1) with parameters:

$$ \mu_1 = 0.2763, \sigma_1 = 0.18, v_1 = -0.15, S_0^1 = 100, $$
$$ \mu_2 = 0.28, \sigma_2 = 0.19, v_2 = -0.3,\ S_0^2 = 100, $$
$$ r = 0.05, \lambda = 0.17, k = 0.5\%, K = \exp(0.1T)\, S_0^1. $$

We fix $\alpha = 0.95$. Time of maturity and $CVaR_{0.95}$ upper bound will vary in this example. The method discussed in Sections 3.2 will be used to derive total CVaR hedging costs X_0. By equation (4.2) survival probabilities of target clients can be derived, and then using survival probabilities from mortality Table the target clients' ages can be found. Table 4.1 and 4.2 show the total CVaR hedging costs for H with different maturities and revision periods under $\mathrm{CVaR}_{0.95}$ threshold 3.5, 5.

Tables 4.1 and 4.2 show that when transactions occur more frequently, both expected total hedging errors and transaction costs are increased. However, the difference between them is decreased, which means, with frequent transactions, the hedging errors can almost offset the transaction costs during the trading. And the total costs will be bigger for long maturity contracts. It is not surprising that the larger the CVaR threshold is, the smaller the hedging costs would be.

Maturity T (years)	Revision period	HE	TC	HE-TC	Hedging price	Total Cost
T=1	Monthly	-0.004	0.5535	-0.5575	4.4595	5.017
	Biweekly	0.5683	0.7968	-0.2285	4.6712	4.8997
	Weekly	0.9627	1.0918	-0.1291	4.9384	5.0675
T=3	Monthly	-0.0278	0.9185	-0.9463	7.2557	8.202
	Biweekly	0.8056	1.2829	-0.4773	7.6245	8.1018
	Weekly	1.5288	1.7277	-0.1989	8.0902	8.2891
T=5	Monthly	-0.0188	1.1306	-1.1494	8.4652	9.6146
	Biweekly	1.0257	1.5722	-0.5465	8.9451	9.4916
	Weekly	1.8168	2.1223	-0.3055	9.5518	9.8573
T=10	Monthly	-0.0755	1.8716	-1.9471	7.9546	9.9017
	Biweekly	1.4848	2.5936	-1.1088	8.5986	9.7074
	Weekly	2.6345	3.4845	-0.85	9.423	10.273

Table 4.1 The total CVaR hedging costs in the Jump-Diffusion market model subject to $\mathrm{CVaR}_{0.95}$ constrain 3.5.

Utilizing formula (4.2), we can find the values of the corresponding clients' survival probabilities ${}_{T}p_{x}$, then, based on the most recently published 2014 United States life Table, the target clients' ages can be derived. Results are presented in Table 4.3.

From Table 4.3 we can see that with more frequent transactions, the corresponding age of target clients would decrease for around 0-2 years. It is interesting to notice that, with the time to maturity increases, the corresponding survival probability increases first, but then decreases. Furthermore, we would like to compare the target clients' ages of the Jump-Diffusion model with that of the Black-Scholes model. Table 4.4 and 4.5 show the total hedging costs in the Black-Scholes model with Leland adjusted volatility $\hat{\sigma} = \sigma\hat{}2(1 + 2k \,/\, \sigma \;\; \sqrt{}(2 \,/\, \pi\Delta t))$, which consists a risk-free asset and a stock with following parameters:

$$\mu = 0.2763, \sigma = 0.18, S_0 = 100,\ r = 0.05.$$

Maturity T (years)	Revision period	HE	TC	HE-TC	Hedging price	Total Cost
T=1	Monthly	-0.0421	0.5501	-0.5922	4.8781	5.4703
	Biweekly	0.4883	0.7757	-0.2874	5.0947	5.3821
	Weekly	0.9254	1.0798	-0.1544	5.3674	5.5218
T=3	Monthly	-0.114	0.8956	-1.0096	7.5697	8.5793
	Biweekly	0.7949	1.2446	-0.4497	7.9424	8.3921
	Weekly	1.4997	1.6939	-0.1942	8.4125	8.6067
T=5	Monthly	-0.0098	1.1102	-1.12	8.6937	9.8137
	Biweekly	1.0007	1.5727	-0.572	9.1782	9.7502
	Weekly	1.7486	2.1087	-0.3601	9.7903	10.1504
T=10	Monthly	-0.0428	1.8037	-1.8465	8.3098	10.1563
	Biweekly	1.5052	2.4742	-0.969	8.9579	9.9269
	Weekly	2.6391	3.3543	-0.7152	9.7863	10.5015

Table 4.2 The total CVaR hedging costs in the Jump-Diffusion market model subject to $\mathrm{CVaR}_{0.95}$ constrain 5.

		$\mathrm{CVaR}_{0.95} \le 3.5$		$\mathrm{CVaR}_{0.95} \le 5$	
Maturity T (years)	Revision period	${}_{T}p_{x}$	age	${}_{T}p_{x}$	age
T=1	Monthly	0.835964363	91	0.771252882	95
	Biweekly	0.830930031	91	0.763609444	95
	Weekly	0.832373602	91	0.7656106	95
T=3	Monthly	0.921118746	72	0.887163068	76
	Biweekly	0.915598372	73	0.882520179	77
	Weekly	0.91612292	73	0.881678456	77
T=5	Monthly	0.945680035	61	0.925619994	65
	Biweekly	0.946061071	60	0.923352303	66
	Weekly	0.949869457	59	0.925099011	65
T=10	Monthly	0.873472372	60	0.852242994	63
	Biweekly	0.873024528	60	0.848519283	63
	Weekly	0.882612496	59	0.866824737	60

Table 4.3 Survival probabilities and ages for insureds in the Jump-Diffusion market model.

Maturity T (years)	Revision period	HE	TC	HE-TC	Hedging price	Total Cost
T=1	Monthly	0.075	0.5541	-0.4791	4.4927	4.9718
	Biweekly	0.5785	0.8016	-0.2231	4.7197	4.9428
	Weekly	0.9745	1.1049	-0.1304	5.0048	5.1352
T=3	Monthly	0.1212	0.9421	-0.8209	6.9094	7.7303
	Biweekly	0.9835	1.3174	-0.3339	7.2985	7.6324
	Weekly	1.5946	1.7946	-0.2	7.7883	7.9883
T=5	Monthly	0.1498	1.1872	-1.0374	7.89	8.9274
	Biweekly	1.1433	1.6411	-0.4978	8.3837	8.8815
	Weekly	1.99	2.2548	-0.2648	9.007	9.2718
T=10	Monthly	0.1792	1.7785	-1.5993	7.8601	9.4594
	Biweekly	1.6731	2.505	-0.8319	8.5287	9.3606
	Weekly	2.8986	3.4248	-0.5262	9.3815	9.9077

Table 4.4 The total CVaR hedging costs in the Black-Scholes model subject to $\mathrm{CVaR}_{0.95}$ constrain 3.5.

Maturity T (years)	Revision period	HE	TC	HE-TC	Hedging price	Total Cost
T=1	Monthly	0.0553	0.5621	-0.5068	4.0775	4.5843
	Biweekly	0.6064	0.8154	-0.209	4.2993	4.5083
	Weekly	1.0048	1.1249	-0.1201	4.5783	4.6984
T=3	Monthly	0.1354	0.9653	-0.8299	6.6002	7.4301
	Biweekly	0.942	1.3438	-0.4018	6.9849	7.3867
	Weekly	1.6206	1.8341	-0.2135	7.4696	7.6831
T=5	Monthly	0.1839	1.1961	-1.0122	7.6601	8.6723
	Biweekly	1.1605	1.6724	-0.5119	8.1494	8.6613
	Weekly	2.0469	2.3036	-0.2567	8.7677	9.0244
T=10	Monthly	0.238	1.863	-1.625	7.6035	9.2285
	Biweekly	1.672	2.5877	-0.9157	8.2699	9.1856
	Weekly	2.9041	3.539	-0.6349	9.121	9.7559

Table 4.5 The total CVaR hedging costs in the Black-Scholes model subject to $\mathrm{CVaR}_{0.95}$ constrain 5.

The corresponding survival probabilities and ages for target clients are present in Table 4.6.

Results show that the total costs to hedge the contract in the Jump-Diffusion model are higher than the costs required to hedge the same contract in the Black-Sholes model. And the corresponding target clients are slightly younger in the Black-Sholes model.

		$CVaR_{0.95} \leq 3.5$		$CVaR_{0.95} \leq 5$	
Maturity T (years)	Revision period	${}_Tp_x$	age	${}_Tp_x$	age
T=1	Monthly	0.820957	92	0.755177	96
	Biweekly	0.817059	92	0.747088	96
	Weekly	0.826512	92	0.752021	96
T=3	Monthly	0.913056	74	0.878013	77
	Biweekly	0.903189	75	0.872782	78
	Weekly	0.910379	74	0.874859	77
T=5	Monthly	0.944219	61	0.921957	66
	Biweekly	0.941325	62	0.917297	66
	Weekly	0.941998	62	0.917328	66
T=10	Monthly	0.913881	55	0.893732	58
	Biweekly	0.912962	55	0.897741	57
	Weekly	0.920473	54	0.906017	56

Table 4.6 Survival probabilities and ages for insureds in the Black-Scholes model.

4.2 CVaR-hedging in the area of financial regulation

Let us pay attention to another application of CVaR-hedging, which was confirmed by the most recent Basse III as a necessary shift from VaR to CVaR to make a regulation in financial sphere more realistic.

Let γ be the amount of required regulatory capital per unit of CVaR. With the initial wealth $v \leq V_0$, where V_0 is the minimal initial wealth required to hedge perfectly, a claim is partially hedged by using CVaR-optimal strategy as discussed in Section 3.3. Denote $CVaR_\alpha(v)$ as the value of CVaR when such a hedging strategy is imposed. The option seller compares the required regulatory captial $\rho_\gamma^\alpha(0)= \gamma\ CVaR_\alpha(0)$ (no hedging) with $\rho_\gamma^\alpha(v)= \gamma\ CVaR_\alpha(v) + v$ (hedging with initial wealth v).

The ratio

$$R_\alpha(v) = \frac{\rho_\gamma^\alpha(v)}{\rho_\gamma^\alpha(0)}, \tag{4.3}$$

measures the relative attractiveness of CVaR partial hedging. More specifically, $R_\alpha(v) \le 1$ means by imposing CVaR-hedging with initial wealth v, the option seller can reduce both required regulatory capital and the risk exposure CVaR-wise.

Melnikov and Nosrati (2017) provided a Figure that shows the aforesaid ratio for varying γ levels and initial capital v in the Black-Scholes model. Here, we would like to extend their analysis to the Jump-Diffusion model with transaction costs. Let us take the following parameters of the model (diffusion part parameters are the same as Melnikov and Nosrati (2017)):

$$\sigma = 0.3, \mu = 0.135, r = 0.05, S_0 = 100, v = -0.15, \lambda = 0.17,$$
$$\alpha = 0.99, k = 0.5\%.$$

We consider a European call option with strike price $K = 110$, and maturity time $T = 0.25$. Figure 4.1 shows $R_\alpha(v)$ for different γ and revision periods.

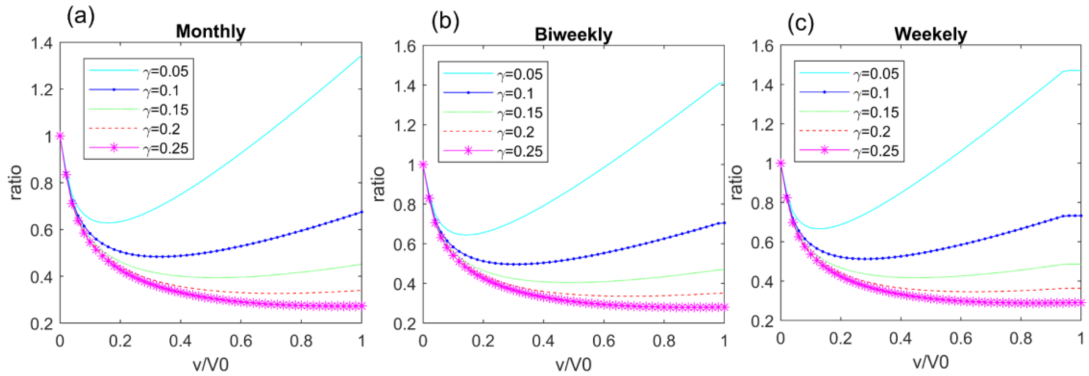

Figure 4.1 $R_\alpha(v)$ for different γ and revision period.

For all three different hedging frequencies, only when $\gamma = 0.05$ (small γ), ratio $R_\alpha(v)$ would be larger than 1 for large v, which means when required regulatory capital per unit of CVaR is small, it may not be optimal to spend large amount of money to hedge. Otherwise, CVaR hedging is always preferable than no hedging. Moreover, for a fixed level of γ and $v, R_\alpha(v)$ would increase with more frequent transactions. Finally, compared with the plot in Melnikov and Nosrati (2017), Figure 4.1 shows that, for each revision period, $R_\alpha(v)$ is always lower in the Jump-Diffusion model than in the Black-Scholes model at the same level of γ and initial investment.

5 Conclusion

In this paper, we consider a Jump-Diffusion market model with proportional transaction costs. The option pricing formula in this market is derived. The problem of

CVaR partial hedging is studied by constructing two kinds of hedging strategies: the one minimizes initial costs under a CVaR constrain and the one minimizes CVaR with an initial wealth constrain. Moreover, applications of CVaR-hedging are discussed. CVaR-hedging is implemented to derive the age of target clients for an embedded call option in the Jump-Diffusion model. Results show that the target clients in the Jump-Diffusion market model would be older than in the Black-Sholes model for the same contract. Meanwhile, the application of CVaR-hedging in the area of financial regulation is investigated. Plots for attractive CVaR-hedging region are given for different required regulatory capital levels and hedging frequencies. Those plots show that CVaR hedging would be more attractive when required regulatory capital level is high and the time between two rebalance is long. Also, CVaR hedging is preferable in the Jump-Diffusion model than in the Black-Sholes model.

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Appendix A

Proof for Lemma 2.1 and Proposition 2.2.

Consider a portfolio constructed by a short position in an option, φ_t^1 shares of long position in S^1, φ_t^2 shares of long position in S^2. The time t value of this portfolio is:

$$\Pi_t = \varphi_t^1 S_t^1 + \varphi_t^2 S_t^1 - \bar{V}(t), \quad \text{(A.1)}$$

Where $\bar{V}(t)$ is the value for that option at time t.

From t to $t + \Delta$t, the value of the portfolio changes by amount:

$$\Delta\Pi_t = \varphi_t^1 \Delta S_t^1 + \varphi_t^2 \Delta S_t^2 - \Delta\bar{V} \text{-} k|w_t^1|S_t^1 \text{-} k|w_t^2|S_t^2, \quad \text{(A.2)}$$

where $\Delta S_t^i \quad (i = 1,2)$ represents the corresponding price changes of underlying assets, and w_t^i represents the corresponding changes of shares of underlying assets between time t to $t + \Delta$t.

The number of jumps during the time interval $[t, t + \Delta$t) satisfies:

$$\Delta N_t = N_{t+\Delta t} - N_t = \begin{cases} 0 \quad with \ probability \ 1 - \lambda\Delta t - o(\Delta t), \\ 1 \quad with \ probability \ \lambda\Delta t + o(\Delta t), \\ others, \quad with \ probability \ o(\Delta t). \end{cases} \quad \text{(A.3)}$$

So, when t is small, it is reasonable to assume there is at most 1 jump in the time interval $[t, t+\Delta \mathrm{t})$ and the jump can only happen at time t.

1) If $N_{t+\Delta t} - N_t = 0$, by $It\hat{o}$ formula:

$$\Delta S_t^i = S_t^i(\mu_i \Delta t + \sigma_i \Delta W_t), \quad (i = 1,2), \tag{A.4}$$

$$\Delta \bar{V}(\mathrm{t}, S_t^1, S_t^2) = \bar{V}_t \Delta \mathrm{t} + \bar{V}_{s^1} \Delta S_t^1 + \bar{V}_{s^2} \Delta S_t^2 + \frac{1}{2}\sum_{i,j=1}^{2} \sigma_i \sigma_j S_t^i S_t^j \Delta \mathrm{t}. \tag{A.5}$$

Using the first order Taylor formula,

$$w_t^i = \Delta \varphi_t^i = \frac{\partial \varphi_t^i}{\partial S_t^i} \Delta S_t^i + o(\Delta t),$$

$$= \frac{\partial \varphi_t^i}{\partial S_t^i} S_t^i(\mu_i \Delta t + \sigma_i \Delta W_t) + o(\Delta t)$$

$$= \frac{\partial \varphi_t^i}{\partial S_t^i} S_t^i \sigma_i \Delta W_t + \ o(\Delta t) \ \ (i = 1,2). \tag{A.6}$$

Therefore,

$$\Delta \Pi_{t|\Delta N_t = 0} = (\varphi_t^1 - \bar{V}_{s^1})\Delta S_t^1 + (\varphi_t^2 - \bar{V}_{s^2})\Delta S_t^2 - (\bar{V}_t + \frac{1}{2}\sum_{i,j=1}^{2} \sigma_i \sigma_j S_t^i S_t^j)\Delta \mathrm{t}$$
$$-k|w_t^1|S_t^1 \text{-} k|w_t^2|S_t^2. \tag{A.7}$$

2) If $N_{t+\Delta t} - N_t = 1$, we can get:

$$\Delta \Pi_{t|\Delta N_t = 1} = (\varphi_t^1 - \bar{V}_{s^1})\Delta S_t^1 + (\varphi_t^2 - \bar{V}_{s^2})\Delta S_t^2 - (\bar{V}_t + \frac{1}{2}\sum_{i,j=1}^{2} \sigma_i \sigma_j S_t^i S_t^j)\Delta \mathrm{t}$$
$$-v_1 \varphi_t^1 - v_2 \varphi_t^2 - (\bar{V}_+ - \bar{V}) \text{-} k|w_t^1|S_t^1 \text{-} k|w_t^2|S_t^2, \tag{A.8}$$

$$w_t^i = \Delta \varphi_t^i = \frac{\partial \varphi_t^i}{\partial S_t^i} \Delta S_t^i + \varphi^i(t, (1-v_1)S_t^1, (1-v_2)S_t^2) - \varphi^i(t, S_t^1, S_t^2) + o(\Delta t)$$

$$= \frac{\partial \varphi_t^i}{\partial S_t^i} S_t^i \sigma_i \Delta W_t + \varphi^i(t, (1-v_1)S_t^1, (1-v_2)S_t^2) - \varphi^i(t, S_t^1, S_t^2) + o(\Delta t)$$
$$(i = 1,2). \tag{A.9}$$

In order to eliminate diversifiable risk, let:

$$\varphi_t^i = \frac{\partial \bar{V}}{\partial S_t^i}, \ \ (i = 1,2). \tag{A.10}$$

Consider the expected value of transaction costs under measure P:

$$E\left(k\left|w_t^i\right|S_t^i|\mathcal{F}_t\right) = \ kS_t^i E\left(\left|w_t^i\right|\right)$$
$$= kS_t^i\left[E\left(\left|w_t^i\right| | \Delta N_t = 0\right)P(\Delta N_t = 0) + E\left(\left|w_t^i\right| | \Delta N_t = 1\right)P(\Delta N_t = 1)\right]$$
$$= (1 - \lambda \Delta t)\sigma_i k {S_t^i}^2 E\left(\left|\bar{V}_{s^i s^i} \Delta W_t\right|\right)$$

$$+\lambda\Delta t k S_t^i\left\{\sigma_i S_t^i E\left(\left|\bar{V}_{s^i s^i}\Delta W_t\right|\right)+E\left(\left|\frac{\partial\bar{V}_+}{\partial S_t^1}-\frac{\partial\bar{V}}{\partial S_t^1}\right|\right)\right\}$$

$$=\sigma_i k {S_t^i}^2\left|\bar{V}_{s^i s^i}\right|E(|\Delta W_t|)+k\lambda S_t^i E\left(\left|\frac{\partial\bar{V}_+}{\partial S_t^1}-\frac{\partial\bar{V}}{\partial S_t^1}\right|\right)\Delta t$$

$$\approx\sigma_i k {S_t^i}^2\left|\bar{V}_{s^i s^i}\right|\sqrt{\frac{2}{\pi\Delta t}}\Delta t-k\lambda {S_t^i}^2 \upsilon_i\left|\bar{V}_{s^i s^i}\right|\Delta t. \tag{A.11}$$

The expected return of the portfolio should be equal to the risk-free rate, which means:

$$E(\Delta\Pi_t|\mathcal{F}_t)=E(r\Pi_t\Delta\mathrm{t}|\mathcal{F}_t) \tag{A.12}$$

By (A.1), (A.7) and (A.8) and (A.11),

$$E(\Delta\Pi_t|\mathcal{F}_t)=(1-\lambda\Delta t)\Delta\Pi_{t|\Delta N_t=0}+\lambda\Delta t\Delta\Pi_{t|\Delta N_t=1}$$

$$=-\left(\bar{V}_t+\frac{1}{2}\sum_{i,j=1}^{2}\sigma_i\sigma_j S_t^i S_t^j\right)\Delta\mathrm{t}-\lambda\Delta\mathrm{t}\sum_{i=1}^{2}\upsilon_i S_t^i\bar{V}_{s^i}-\lambda\Delta\mathrm{t}(\bar{V}_+-\bar{V})$$

$$-\left[\sigma_i k {S_t^i}^2\left|\bar{V}_{s^i s^i}\right|\sqrt{\frac{2}{\pi\Delta t}}\Delta t-k\lambda {S_t^i}^2\upsilon_i\left|\bar{V}_{s^i s^i}\right|\Delta t\right]. \tag{A.13}$$

$$E(r\Pi_t\Delta\mathrm{t}|\mathcal{F}_t)=r\Pi_t\Delta\mathrm{t}=r\left(\sum_{i=1}^{2}V_{s^i}S_t^i-\bar{V}(t)\right)\Delta\mathrm{t}. \tag{A.14}$$

Let (A.13)=(A.14) , after some simple calculation, we can get the formula in Lemma 2.1.

Unfortunately, the PDE is nonlinear, so there is no specific formula for option pricing with transaction costs, only a numerical result can be got.

However, in our paper, we consider a European call option based on the first asset $H=(S_T^1-K)^+$. This option is constructed only by the first stock, so its value process $\bar{V}(t)$ should be independent from S_t^2. i.e $\bar{V}_{s^2}=0$. Therefore, we can simplify Lemma 2.1, and get the value process for H, denoted as $\bar{V}(t)$, should satisfy:

$$\bar{V}_t+\bar{V}_{s^1}S_t^1(r+\upsilon_1\lambda)+\frac{1}{2}\bar{V}_{s^1 s^1}{\hat{\sigma}_1}^2{S_t^1}^2-r\bar{V}+(\bar{V}^+-\bar{V})\lambda=0. \tag{A.15}$$

Also notice that, if transaction costs do not exist, the market would be complete. Any option would have a unique value process. By extended It$\hat{o}$ formula, option value process for H should satisfy:

$$dV(t)=V_t dt+V_{s^1}S_t^1\mu_1 dt+V_{s^1}S_t^1\sigma_1 dW_t+\frac{1}{2}V_{s^1 s^1}{\sigma_1}^2{S_t^1}^2 dt+(V^+-V)dN_t$$

$$= \left[V_t + V_{s^1}S_t^1(r + v_1\lambda^*) + \frac{1}{2}V_{s^1s^1}{\sigma_1}^2{S_t^1}^2 + (V^+ - V)\lambda^*\right]dt$$

$$+V_{s^1}S_t^1\sigma_1 dW_t^* + (V^+ - V)dM_t^Q, \qquad (A.16)$$

where $M_t^Q = N_t - \lambda^* t$ and W_t^* are martingales under measure Q.

Using integration by parts:

$$de^{-rt}V(t) = V(t)de^{-rt} + e^{-rt}dV(t)$$

$$= e^{-rt}\left[V_t + V_{s^1}S_t^1(r + v_1\lambda^*) + \frac{1}{2}V_{s^1s^1}{\sigma_1}^2{S_t^1}^2 - rV + (V^+ - V)\lambda^*\right]dt$$

$$+e^{-rt}V_{s^1}S_t^1\sigma_1 dW_t^* + e^{-rt}(V^+ - V)dM_t^Q \qquad (A.17)$$

Since $e^{-rt}V(t)$ is a martingale under Q, the drift term of the equation (A.17) should be vanished, which means option value should satisfy:

$$V_t + V_{s^1}S_t^1(r + v_1\lambda^*) + \tfrac{1}{2}V_{s^1s^1}{\sigma_1}^2{S_t^1}^2 - rV + (V^+ - V)\lambda^* = 0. \qquad (A.18)$$

Both $(A.15)$ and (A.18) satisfy the boundary condition $\bar{V}(T) = V(T) = (S_T^1 - K)^+$. Thus, the price process of H with transaction costs is just the price process of this option in the complete market with adjusted volatility $\hat{\sigma}_1$ and poisson intensity λ, written as:

$$\bar{V}(t) = V(t, S_t^1, \hat{\sigma}_1, \lambda). \qquad (A.19)$$

Moreover, for call option $\bar{V}_{s^1s^1} \geq 0$.

Appendix B

Proof for Lemma 3.2.

By Kirch(2005), in the Jump-Diffusion model, the density for the martingale measure can be represented in terms of S_T^1 and N_t:

$$\frac{dP^*}{dP} = exp\left(\alpha^* W_t - \frac{{\alpha^*}^2}{2}t + (\lambda - \lambda^*)t + (ln\lambda^* - ln\lambda)\mathrm{N}_t\right)$$

$$= \left(S_0^1\, exp\left\{\sigma_1 W_t + \left(\mu_1 - \frac{1}{2}{\sigma_1}^2\right)T\right\}(1 - v_1)^{\mathrm{N}_T}\right)^{\frac{\alpha^*}{\sigma_1}}$$

$$\times \frac{1}{{S_0^1}^{\frac{\alpha^*}{\sigma_1}}} exp\left(-\frac{\alpha^*\mu_1}{\sigma_1}T + \frac{\sigma_1\alpha^*}{2}T - \frac{{\alpha^*}^2}{2}T + (\lambda - \lambda^*)T\right) \times \left(\frac{\lambda^*}{\lambda(1 - v_1)^{\frac{\alpha^*}{\sigma_1}}}\right)^{\mathrm{N}_T}$$

$$= g({S_T^1})^{\frac{\alpha^*}{\sigma_1}} \qquad (B.1)$$

where $b = \frac{\lambda^*}{\lambda(1-v_1)^{\frac{\alpha^*}{\sigma_1}}}$ and $g = \frac{1}{{S_0^1}^{\frac{\alpha^*}{\sigma_1}}} exp\left(-\frac{\alpha^*\mu_1}{\sigma_1}T + \frac{\sigma_1\alpha^*}{2}T - \frac{{\alpha^*}^2}{2}T + (\lambda - \lambda^*)T\right)$.

For illustration purpose, we assume $\alpha^* < 0$, and $b \geq 1$, therefore,

$$\left\{\frac{dP^*}{dP} > a\right\} = \left\{g(S_T^1)^{\frac{\alpha^*}{\sigma_1}} b^{\mathrm{N}_t} > a\right\} = \left\{S_T^1 < mb^{*\mathrm{N}_t}\right\}, \qquad (B.2)$$

$$\left\{\frac{dP^*}{dP} = a\right\} = 0, \qquad (B.3)$$

where $b^* = b^{-\frac{\sigma_1}{\alpha^*}}$.

Equation (3.4) -(3.6) become:

$$\tilde{\varphi}(z) = 1_{\left\{\frac{dP^*}{dP} > \tilde{a}(z)\right\}} = 1_{\left\{S_T^1 < m(z) b^{*\mathrm{N}_t}\right\}},$$

$$m(z) = sup\left\{m \geq 0\text{: } E_P\left[(H-z)^+ 1_{\left\{S_T^1 < mb^{*\mathrm{N}_t}\right\}}\right] \leq (\bar{C} - Z)(1-\alpha)\right\},$$

$$\gamma(z) = 0.$$

Notice that since $z \geq 0$,

$$(H-z)^+ = ((S_t^1 - k)^+ - z)^+ = \left(S_t^1 - (k+z)\right)^+. \qquad (B.4)$$

Now, consider $E_P\left[(H-z)^+ 1_{\left\{S_T^1 < mb^{*\mathrm{N}_t}\right\}}\right]$.

$$E_P\left[(H-z)^+ 1_{\left\{S_T^1 < mb^{*\mathrm{N}_t}\right\}}\right] = E_P\left[E_P[(H-z)^+ 1_{\left\{S_T^1 < mb^{*\mathrm{N}_t}\right\}}] | \mathrm{N}_t\right]$$

$$= \sum_{n=0}^{\infty} E_p\left(\left(\tilde{S}_{0,n}^1 - K(z)\right)^+ 1_{\left\{\tilde{S}_{0,n}^1 < mb^{*n}\right\}} \middle| \Pi_t = n\right) p_{n,T}$$

$$= \sum_{n=0}^{\infty} E_p\left(\left(\tilde{S}_{0,n}^1 - K(z)\right)^+ 1_{\left\{\tilde{S}_{0,n}^1 < mb^{*n}\right\}}\right) p_{n,T}. \qquad (B.5)$$

Denote $\delta(n) = \tilde{S}_{0,n}^1 \, exp(\mu_1 T) \wedge_+ (K(z), mb^{*n}) - K(z) \wedge_- (K(z), mb^{*n})$, (B.6)

$$n_a(m,z) = inf\{n, mb^{*n} \geq K(z)\}. \qquad (B.7)$$

Then,

$$E_P\left(\left(\tilde{S}_{0,n}^1 - K(z)\right)^+ 1_{\left\{\tilde{S}_{0,n}^1 < mb^{*n}\right\}}\right) = \begin{cases} 0 & if \ \ mb^{*n} < K(z), \\ \delta(n) & if \ \ mb^{*n} > K(z). \end{cases} \qquad (B.8)$$

$$E_P\left[(H-z)^+ 1_{\left\{\tilde{S}_{0,n}^1 < mb^{*\mathrm{N}_t}\right\}}\right] = \sum_{n=\ n_a(m,z)}^{\infty} E_P\left(\left(\tilde{S}_{0,n}^1 - K(z)\right)^+ 1_{\left\{\tilde{S}_{0,n}^1 < mb^{*n}\right\}}\right) p_{n,T}$$

$$= \sum_{n=\ n_a(m,z)}^{\infty} \delta(n)\, p_{n,T}. \qquad (B.9)$$

Since $E_P\left[(H-z)^+ 1_{\left\{S_T^1 < m(z) b^{*\mathrm{N}_t}\right\}}\right]$ is a non-increasing function of m, and

$$E_P\left[(H-z)^+ 1_{\left\{S_T^1<0*b^{*N_t}\right\}}\right]=0,\quad E_P[(H-z)^+]\ge(\bar{C}-Z)(1-\alpha), \tag{B.10}$$

$m(z)$ is the unique solution for the equation:

$$\sum_{n=\ n_a(m,z)}^{\infty}[\tilde{S}_{0,n}^1\, exp(\mu_1 T)\wedge_+ (K(z), mb^{*n}) - K(z)\wedge_- (K(z), mb^{*n})]p_{n,T}$$
$$=(\bar{C}-z)(1-\alpha). \tag{B.11}$$

Also, by the notation $n_a(z)=n_a(m(z),z)=inf\{n\text{: } m(z)b^{*n}\ge K(z)\}$,

$$d(z)=E_{P^*}\left[(H-z)^+\left(1-\tilde{\varphi}(z)\right)\right]=\sum_{n=0}^{\infty}E_{P^*}\left(\left(\hat{S}_{0,n}^1-K(z)\right)^+ 1_{\{\hat{S}_{0,n}^1>m(z)b^{*n}\}}\right)p_{n,T}^*$$

$$=\sum_{n=0}^{n_a(z)-1}E_{P^*}\left(\left(\hat{S}_{0,n}^1-K(z)\right)^+ 1_{\{\hat{S}_{0,n}^1>m(z)b^{*n}\}}\right)p_{n,T}^*$$

$$+\sum_{n=n_a(z)}^{\infty}E_{P^*}\left(\left(\hat{S}_{0,n}^1-K(z)\right)^+ 1_{\{\hat{S}_{0,n}^1>m(z)b^{*n}\}}\right)p_{n,T}^*$$

$$=\sum_{n=0}^{n_a(z)-1}\mathrm{C}^{\mathrm{BS}}\left(\hat{S}_{0,n}^1,K(z),T\right)p_{n,T}^*$$

$$+\sum_{n=n_a(z)}^{\infty}[\hat{S}_{0,n}^1\Phi\left(d_+\left(\hat{S}_{0,n}^1,m(z)b^{*n}\right)\right)-e^{-rT}K(z)\Phi\left(d_-\left(\hat{S}_{0,n}^1,m(z)b^{*n}\right)\right)]p_{n,T.}^*$$

(A. 10)

And $\hat{z}$ is a point of minimum of function $d(z)$, over interval (0, $\bar{C}$).

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