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How do Economic Variables Affect the Pricing of Commodity Derivatives and Insurance?

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How do economic variables affect the pricing of commodity derivatives and insurance?

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Abstract

This paper focuses on designing and pricing commodity derivatives and insurance within a novel financial engineering framework that can be subsequently tested empirically using commodity price data. Optimal contract solutions are obtained and interpreted. We quantify explicitly how derivative prices and insurance premiums are affected by economic variables linked to commodity supply and demand. Our results generalize some existing commodity derivative pricing models and further show under which conditions there will be no trading of derivative instruments and insurance. We report GMM estimates of the model parameters for a large dataset of commodity futures. These results also contribute to a better understanding of the "financialization" of commodities.

Keywords: option pricing; risk management; insurance; derivatives; commodities; demand; elasticity;

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How do economic variables affect the pricing of commodity derivatives and insurance?

1. Introduction

The pricing of insurance contracts and financial derivatives on commodities such as crude oil or wheat are methodologically linked by their use of risk-neutral valuation methods such as the seminal Black-Scholes formula. Unlike financial security prices, which are driven by priced equity risk factors, commodity prices are mainly influenced by commodity-specific economic variables, which depend on fundamental weather, production, and storage variables. Commodity prices thus result from demand, supply, inventory, and economic risk factors. Although a large literature exists that takes a reduced-form approach to model the prices of commodity derivatives and insurance, a deeper understanding of their pricing requires a richer model of these economic fundamentals. Indeed, financial engineering methods are often silent when it comes to quantifying the exact role of economic variables on these contingent claim prices. This paper aims to make a contribution to the literatures on commodity derivative and insurance contract pricing by proposing an economically-motivated model, which allows us to generate new insights on pricing and risk management.

Generally speaking, there are two approaches to modelling commodity prices prior to modelling their contingent claims. The first approach belongs to the literature on finance and financial engineering, which uses models based on diffusion processes, and begins with Black [1976]. The multifactor models of Gibson and Schwartz [1990] and Nielsen and Schwartz [2004], the CEV model of Geman and Shiy [2009], and the mean reverting models of Ribeiro and Hodges [2004] are just a few examples. The second approach belongs to the economics literature and is based on rational commodity storage. In particular, this paper aims to adapt to derivative and insurance pricing the framework found in the Deaton and Laroque models [1992, 1995, 1996].[2] By making explicit the price elasticity parameter, this paper also relates to the CEV option pricing model in Cox [1975].

This paper therefore aims to combine the two approaches mentioned above by taking an established methodology from the rational expectations theory of storage, and then modifying and applying it to a financial engineering framework. As a result, it develops a new methodology that directly quantifies the impact of economic variables on commodity insurance and derivative prices. To achieve this goal requires tackling several mathematical and technical problems, which we solve in this paper. The findings described in this paper are also of interest for researchers working on the "financialization" of commodities (Basak and Pavlova [2016], Cheng and Xiong [2014], Tang and Xiong [2012]). Indeed there is great interest in understanding the closer link between commodity and financial markets facilitated by exchange-traded derivatives. The methods and results in this paper should be useful to academics, traders, and practitioners in finance and financial engineering, as well as those in insurance, reinsurance, and risk management.

The remainder of the paper is as follows. In section 2, we develop a representative agent model. Section 3 presents the dynamics for demand and price, defines the loss function and solves for the premium. In section 4, optimal contracts are solved in the general case, and an application is provided for the special case where Value-at-risk is used as the risk measure. Section 5 presents empirical estimates of the parameters of the model, using a large dataset of commodity futures contract prices and GMM estimation. We conclude in section 6.

[2] The details of a storage model have been laid out in a full chapter in Geman [2015].

2. Representative agent model

Let us consider a representative agent with an isoelastic or power utility function given by:

$$u(x) = \frac{x^{1-\phi}}{1-\phi}, \text{if } \phi \neq 1 \text{ and } u(x) = \log(x) \text{ otherwise.}$$

In this model the parameter ϕ represents the coefficient of relative risk aversion (RRA) for the representative consumer/investor.[3] In this paper, we focus our attention on a single good, and treat spending on the other good as a residual that can be added to the agent's utility by using a quasi-linear utility function. Therefore, we consider the agent will be solving the following problem:

$$\max_{x,m} [ku(x) + m], \quad \text{s.t.} \quad px + m = B.$$

where m is residual income for all other goods, k is a constant, p_x is the price of good x and B is the budget constraint. It is straightforward to show that the demand is as follows:

$$x^* = \left(\frac{k\phi}{(1-\phi)p}\right)^{\frac{1}{1-\phi}}$$

If we want to look at the inverse demand, it can be shown that

$$p(x) = \frac{k\phi}{1-\phi}\left(\frac{1}{x^*}\right)^{1-\phi}$$

To simplify the notation, denote $c = \frac{k\phi}{1-\phi}$ and $\gamma = \phi - 1$, then we can consider the following inverse demand function:

$$p(x) = cx^{\gamma}$$

[3] Other functional forms are considered in the appendix.

The form of the demand function for different values of c and γ are depicted in the following graph:

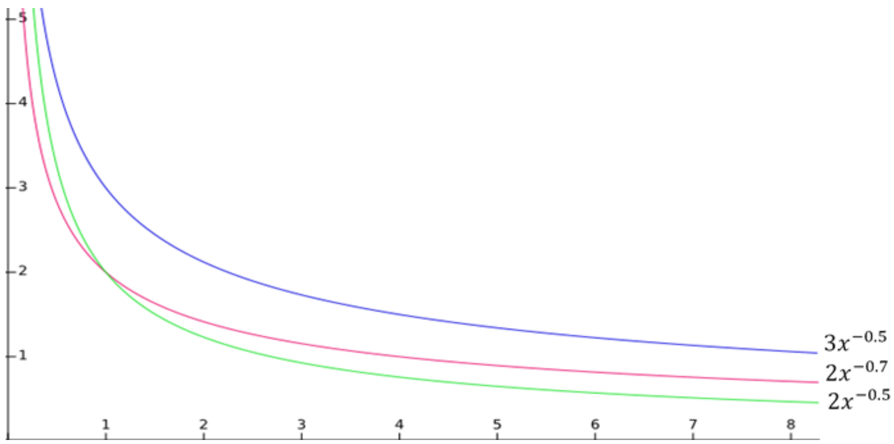

Figure 1: Demand functions for different values of c and γ

As we are focusing our attention only on one good, we can rescale the demand unit and consider $c = 1$. Since prices are non-negative and the demand function is non-increasing, we need to check the two following conditions: $\frac{k\phi}{1-\phi} > 0$, and $\frac{d}{dx}\left(\frac{k\phi}{1-\phi}x^{\phi-1}\right) < 0$, respectively. These conditions imply $0 < \phi < 1$, which implies $-1 < \gamma < 0$.

3. Economic model, loss distribution, and premium

3.1 The demand and price process

Let us consider a stochastic demand process following geometric Brownian motion (gBm) dynamics as follows:

$$\frac{dx_t}{x_t} = \mu dt + \sigma dw_t,$$

where in this case $(w_t)_{0\le t\le T}$ is a standard Brownian motion, μ is the drift term representing the rate of growth in consumption, and σ represents the magnitude of the demand volatility. As a result, the dynamics of the demand process given above can be written as follows:

$$x_t = x_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma w_t}, \qquad t \ge 0$$

This model allows for different markets to be studied, as the demand functions are allowed to vary. This means we can study the effect of both economic and financial market variables on the market demand, and on the resulting derivatives and insurance contracts. Considering the iso-elastic demand function, combining the inverse demand function $p(x)$ with the demand process dynamics yields the following price dynamics for an inverse demand function p:

$$p_t = p(x_t) = cx_t^{\gamma} = p_0 \exp\left(\gamma\left(\mu - \frac{1}{2}\sigma^2\right)t + \gamma\sigma w_t\right)$$

If we consider a new Brownian motion $B_t = -w_t$, one can rewrite the price dynamics as follows:

$$p_t = p_0 \exp\left(\gamma\left(\mu - \frac{1}{2}\sigma^2\right)t - \gamma\sigma B_t\right)$$

This change is necessary because $\gamma < 0$, and as a result $-\gamma\sigma > 0$. Using the Ito formula for the function $f(x,t) = ce^{\gamma\left(\mu-\frac{1}{2}\sigma^2\right)t-\gamma\sigma x}$ gives:

$$\frac{\partial f}{\partial t} = \gamma\left(\mu - \frac{1}{2}\sigma^2\right)f, \qquad \frac{\partial f}{\partial x} = (-\gamma\sigma)f, \qquad \frac{\partial^2 f}{\partial x^2} = \gamma^2\sigma^2 f,$$

resulting in:

$$dp_t = \gamma\left(\mu + \frac{1}{2}(\gamma - 1)\sigma^2\right)p_t dt - \gamma\sigma p_t dB_t.$$

For simplicity, we can write the stochastic differential equation (SDE) of the price process as follows:

$$\frac{dp_t}{p_t} = \nu dt + \eta dB_t, p_0 > 0$$

where $\nu = \gamma\left(\mu + \frac{1}{2}(\gamma - 1)\sigma^2\right)$ and $\eta = -\gamma\sigma$.

It is worth considering some conditions under which the model makes greater economic sense. The first condition is that the drift term of the price, i.e., ν, must be non-negative. Since $\gamma \leq 0$ then this is equivalent to checking that:

$$\mu + \frac{1}{2}(\gamma - 1)\sigma^2 \leq 0.$$

However, on the other hand, the market price of risk needs to be non-negative to make sure that market participants will be involved. For that reason, it is necessary to check whether $\nu - r > 0$. This condition will certainly yield the previous one. The two conditions are economically sensible, but in general they are not necessary to obtain solutions.

3.2 Loss distribution

Let us consider a time horizon T at which we want to introduce a loss variable and write an insurance contract to hedge against the risk of loss. We consider the following non-negative random variable as the loss

$$L = (p_0 - e^{-rT}p_T)_{+.}$$

To motivate this definition, note that if $p_0 - e^{-rT}p_T < 0$ then the excess return i.e., $\log\left(\frac{p_T}{p_0}\right) - rT$, is also negative. Next, we wish to find out the distribution of the loss variable L. First, it is not difficult to see that:

for $x < 0,$ we have $$F_L(x) = 0,$$

for $x = 0$ we have $$F_L(0) = P(p_0 \le e^{-rT} p_T) = N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT}{-\gamma\sigma\sqrt{T}}\right)$$

and for $x > p_0$, we have $$F_L(x) = 1$$

Now let us consider $p_0 \ge x > 0$. In this case, we have:

$$F_L(x) = 1 - P(L > x) = 1 - P((p_0 - e^{-rT} p_T)_+ > x) = 1 - P(p_0 - e^{-rT} p_T > x)$$

$$= 1 - P\left(p_0\left(1 - e^{-rT} e^{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - \gamma\sigma B_T}\right) > x\right)$$

$$= 1 - P\left(e^{rT}\left(1 - \frac{x}{p_0}\right) > e^{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - \gamma\sigma B_T}\right)$$

$$= 1 - P\left(\frac{\log e^{rT}\left(1 - \frac{x}{p_0}\right) - \gamma\left(\mu - \frac{1}{2}\sigma^2\right)T}{-\gamma\sigma\sqrt{T}} > B_1\right)$$

$$= 1 - N\left(\frac{rT + \log\left(1 - \frac{x}{p_0}\right) - \gamma\left(\mu - \frac{1}{2}\sigma^2\right)T}{-\gamma\sigma\sqrt{T}}\right)$$

$$= N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT - \log\left(1 - \frac{x}{p_0}\right)}{-\gamma\sigma\sqrt{T}}\right).$$

In sum, we get:

$$F_L(x) = \begin{cases} 0, & x < 0 \\ N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT - \log\left(1 - \frac{x}{p_0}\right)}{-\gamma\sigma\sqrt{T}}\right), & 0 \le x < p_0. \\ 1, & x \ge p_0 \end{cases}$$

The graph of $F_L(x)$ is depicted as follows:

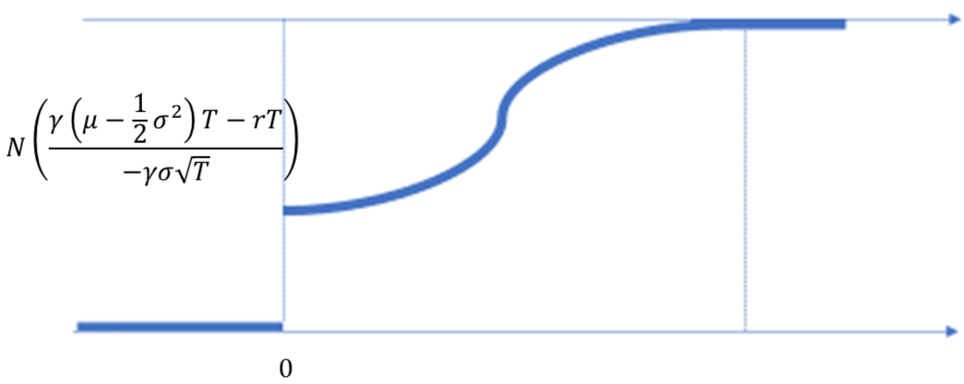

Figure 2: Graph of the distribution function $F_x(L)$ describing the loss function

3.3 Premium

Moreover, we can use risk-neutral valuation principles to price any contract $H = h(p_T)$ by using the risk-free probability measure Q as follows:

$$Price = e^{-rT}E^Q\big(h(p_T)\big) = e^{-rT}E\left(\frac{dQ}{dP}h(p_T)\right),$$

where

$$\frac{dQ}{dP} = e^{\left(\frac{m^2}{2\eta^2}-\frac{m}{2}\right)T}\left(\frac{e^{-rT}p_T}{p_0}\right)^{-\frac{m}{\eta^2}},$$

and $m = \nu - r$. From this, we can show that:

$$\pi(L) = E\left(\frac{dQ}{dP}h(P_T)\right) = \int_0^1 \mathrm{VaR}_t\,\big(h(p_T)\big)d\Gamma(t), \tag{1}$$

where $\Gamma(t) = N\left(N^{-1}(t) - \frac{|m|\sqrt{T}}{\eta}\right)$.

This representation (1) will subsequently help us, along with the risk measures that we use, to find the optimal solutions in a useful way. The above formula can also be used to price options.

To do so, first, we define the contract H based on its payoff profile. Then, we compute the Radon-Nikodym derivative using the estimated underlying price parameters. This is equivalently a risk-neutralization of the process. Finally, we use either analytical formulae or Monte Carlo simulation to compute the price.[4]

4. Designing optimal insurance and pricing derivatives

4.1 Moral hazard issues

In this section, we consider how to use the above model for purposes of pricing optimal insurance contracts as well as options. In the case of designing an optimal insurance contract, we should consider moral hazard and see if we can design a contract that rules it out. The literature on actuarial mathematics deals with this problem in the following manner, by considering that both insurer and insuree should feel the losses.

We assume that a contract X should be such that both X and $(L - X)$, where $L = (p_0 - e^{-rT} p_T)_+$ is the loss, increase when L increases. Therefore, we consider contracts $X = k(L)$, where k belongs to the following set:

$$C = \{k: R_+ \to R_+ | k(x) \text{ and } x - k(x) \text{ are non} - \text{decreasing in } x\}.$$

4.2 Optimal solution

Next, we set up an optimal insurance problem and try to find an optimal solution. For that, we assume that the insuree is a risk-averse agent whose risk is measured according to a distortion risk measure ρ on the set of non-negative random variables defined as follows:

[4] This approach is however not directly applicable to path-dependent options such as American options. For these we would have to consider alternative methods, such as Least-Squares Monte Carlo, which is beyond the scope of this paper.

$$\rho(X) = \int_0^1 \mathrm{VaR}_t(X) d\Pi(t).$$

Here $\Pi: [0,1] \to [0,1]$ is a non-decreasing function so that $\Pi(0) = 0$ and $\Pi(1) = 1$. This family of risk measures includes very important examples, e.g., Value-at-Risk with:

$$\Pi(t) = 1_{[\alpha,1]}$$

or Conditional Value at Risk with:

$$\Pi(t) = \frac{t-\alpha}{1-\alpha} 1_{[\alpha,1]}.$$

A distortion risk measure is a way to better capture the risk by distorting the loss distribution. For instance, some risk measures (e.g., CVaR), distort the distribution by taking a pessimistic point of view towards the risk. Thus, note that the pricing method we propose earlier for the contracts in (1) is also a distortion risk measure, where the prices are distorted according to the pricing kernel distribution and the level of distortion depends on market price of risk. The insuree's global loss is the part of the loss that is not covered by insurer, added up to the amount that is paid for the premium, i.e.,

$$Global\ loss = L - X + \pi(X).$$

Since distortion risk measures are cash-invariant, the risk of the global loss is $\rho(L - X) + \pi(X)$. In order to study insurance premiums, we consider an optimal insurance design problem as proposed in Assa [2015a,b] (or similarly with a budget constraint in Zhuang et al. [2016]):

$$\min_{k \in C} \rho\big(L - k(L)\big) + \delta\pi\big(k(L)\big),$$

for a risk loading factor $\delta \geq 1$ that is used by the insurance company. Using the marginal indemnification function method (MIF) introduced by Assa [2015a] and developed in Assa, Wang and Pantelous [2018], Zhuang et al. [2016] and Assa [2015b], this problem can be re-written as follows:

$$\min_{0\leq k'\leq 1}\int_0^1 \left(\delta\left(1-\Gamma\big(F_L(t)\big)\right)-\left(1-\Pi\big(F_L(t)\big)\right)\right)k'(t)dt,$$

where k' is the derivative of k. The optimal solution is then given by $X = k(L)$, where:

$$k'(t) = \begin{cases} 1, & 1-\Pi\big(F_L(t)\big) > \delta\left(1-\Gamma\big(F_L(t)\big)\right) \\ 0, & 1-\Pi\big(F_L(t)\big) \leq \delta\left(1-\Gamma\big(F_L(t)\big)\right) \end{cases}. \qquad (2)$$

4.3 Solving for the contracts

In this section, we consider a technical assumption, that there are values $a, b \in (0,1)$ such that:

$$1-\Pi(x) > \delta(1-\Gamma(x)) \text{ on } (a,b)$$

and everywhere else than the interval (a,b):

$$1-\Pi(x) < \delta(1-\Gamma(x)) \text{ on } (0,a)\cup(b,1).$$

This assumption holds for many interesting cases including $\rho = \mathrm{VaR}$ and CVaR (see Figure 3).

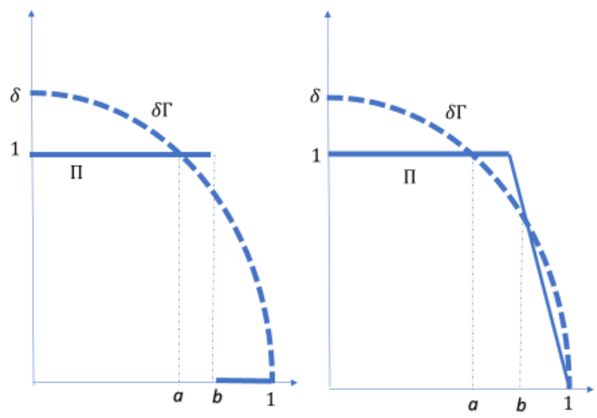

Figure 3: Representing the optimal solution: cases of VaR and cVaR

The existence of the optimal solution (and its form) depends on $F_L(0)$. However, we know that:

$$F_L(0) = N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT}{-\gamma\sigma\sqrt{T}}\right) = N\left(\frac{\left(\mu - \frac{1}{2}\sigma^2\right)\sqrt{T}}{-\sigma} - \frac{r\sqrt{T}}{|\gamma|\sigma}\right).$$

Therefore, increasing the absolute value of γ will decrease the value of $F_L(0)$. The optimal solution in this case either: i) does not exist, ii) is a stop loss policy, or iii) is a two-layer policy. This result can be shown as in the following figure by depicting F_L for different values of γ.

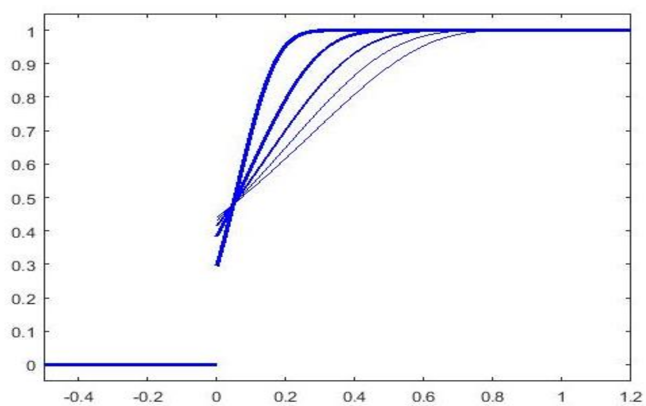

Figure 4: Optimal solution in terms of $F_L(x)$ for different values of γ

We have three cases:

1. If $F_L(0) > b$, or $\frac{r\sqrt{T}}{\sigma\left(N^{-1}(b) - \frac{\left(\mu - \frac{1}{2}\sigma^2\right)\sqrt{T}}{-\sigma}\right)} < \gamma$, then $k' = 0$ and there is no contract.

2. If $a < F_L(0) < b$, or $\frac{r\sqrt{T}}{\sigma\left(N^{-1}(b) - \frac{\left(\mu - \frac{1}{2}\sigma^2\right)\sqrt{T}}{-\sigma}\right)} < \gamma < \frac{r\sqrt{T}}{\sigma\left(N^{-1}(a) - \frac{\left(\mu - \frac{1}{2}\sigma^2\right)\sqrt{T}}{-\sigma}\right)}$, then there is a stop loss policy with retention level that solves

$$N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT - \log\left(1 - \frac{b^*}{p_0}\right)}{-\gamma\sigma\sqrt{T}}\right) = b.$$

   This results in

$$b^* = p_0\left(1 - \exp\left(\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT + \gamma\sigma\sqrt{T}N^{-1}(b)\right)\right).$$

3. Finally, if, $F_L(0) < a$ or $\gamma < \frac{r\sqrt{T}}{\sigma\left(N^{-1}(a) - \frac{\left(\mu - \frac{1}{2}\sigma^2\right)\sqrt{T}}{-\sigma}\right)}$, then the contract is a two-layer policy with upper retention level b^* given above and lower retention level a^* given as

$$a^* = p_0\left(1 - \exp\left(\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT + \gamma\sigma\sqrt{T}N^{-1}(a)\right)\right).$$

   These three cases are represented in **figure 5.**

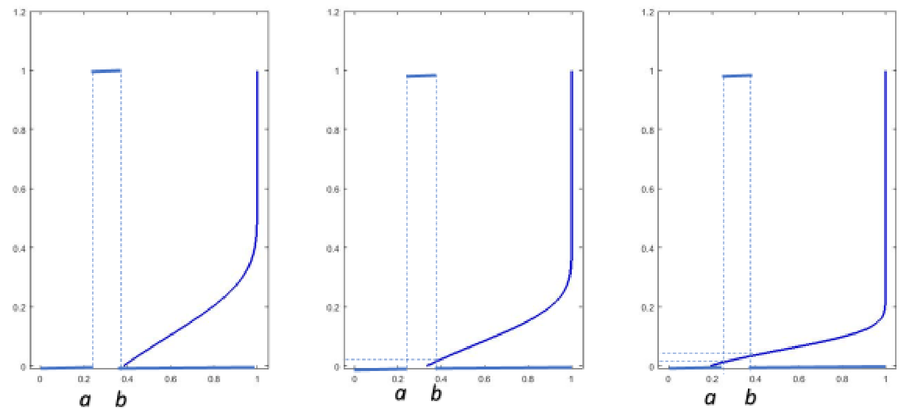

Figure 5: Cases 1 through 3, in terms of values of a and b

Remark 1. Some explanation is warranted here. In case 1, one can see that the probability of no loss, i.e., $F_L(0)$, happens to be large enough ($F_L(0) > b$), and for that reason the insuree does not look at the insurance as a necessary risk management tool, whereas in case 2 there is a demand for insurance from the insuree side. In the most extreme case 3, since the probability of no loss is small ($F_L(0) < a$), then the insurance company needs to include a stop-loss policy to contract to limit its exposure to large losses.

Taking the integral from the marginal indemnity function if (2), we get the indemnity as follows:

$$k(t) = \begin{cases} 0, & t \leq a^* \\ t - a^*, & a^* \leq t \leq b^* \\ b - a^*, & t \geq b^* \end{cases}. \quad (3)$$

from which we can get the following contract:

$$X = \begin{cases} 0, & L \leq a^* \\ L - a^*, & a^* \leq L \leq b^* \\ b - a^*, & L \geq b^* \end{cases}. \quad (4)$$

Now using the particular loss variable we have introduced in this paper (sec. 3.2), we find that:

$$X = \begin{cases} 0, & p_0 - e^{-rT}p_T < 0 \; or \; (p_0 - e^{-rT}p_T \geq 0 \; and \; p_0 - e^{-rT}p_T \leq a^*) \\ p_0 - e^{-rT}p_T - a^*, & a^* \leq p_0 - e^{-rT}p_T \leq b^* \\ b^* - a^*, & p_0 - e^{-rT}p_T \geq b^* \end{cases} . \tag{5}$$

Furthermore, one can easily see that this results in:

$$X = e^{-rT}(e^{rT}(p_0 - a^*) - p_T)_+ - e^{-rT}(e^{rT}(p_0 - b^*) - p_T)_+ \tag{6}$$

Finally, using the pricing kernel of the Black-Scholes model (eq. (1)) we get:

$$Price = P(p_0, e^{rT}(p_0 - a^*), r, T, -\gamma\sigma) - P(p_0, e^{rT}(p_0 - b^*), r, T, -\gamma\sigma) \tag{7}$$

where $P(p_0, K, r, T, \sigma)$ is the price of a put option with risk-free r, volatility σ, expiration T and strike price K.

Remark 2. As one can see, the price of the optimal product is a function of multiple parameters, including γ, the demand elasticity parameter, and σ, the demand volatility. It is clear that the prices of both put options in (7) increase if the absolute value of γ and σ increases. However, the retention levels, a^*, b^* are also functions of these two parameters, which makes it ultimately unclear how the increase in γ and σ will affect the optimal price. We will discuss it within an example when we use Value-at-Risk as the risk measure.

4.4 Example: Value at risk (VaR)

Based on general case that we discussed above, let a be the solution to $\delta\big(1 - \Gamma(a)\big) = 1$ or $a = \Gamma^{-1}\left(1 - \frac{1}{\delta}\right)$. This means:

$$\delta\left(1 - N\left(N^{-1}(a) - \frac{|m|\sqrt{T}}{\eta}\right)\right) = 1$$

or

$$a = N\left(\frac{|m|\sqrt{T}}{\eta} + N^{-1}\left(1 - \frac{1}{\delta}\right)\right).$$

It is also clear that in this case $b = \alpha$.

There are three cases:

1. If $N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT}{-\gamma\sigma\sqrt{T}}\right) \geq \alpha$.

   This is equivalent to $\gamma\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha)\right) \geq rT$. Since $\gamma \leq 0$, therefore,

   $\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha) \leq 0$, and as a result we have to check if

$$\gamma \leq \frac{rT}{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha)}.$$

   In this case, $k' = 0$ and there is no insurance contract.

2. If $a \leq N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT}{-\gamma\sigma\sqrt{T}}\right) < \alpha$.

   This is equivalent to $N\left(\frac{|m|\sqrt{T}}{\eta} + N^{-1}\left(1 - \frac{1}{\delta}\right)\right) \leq N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT}{-\gamma\sigma\sqrt{T}}\right) < \alpha$.

   First, let us look to the right inequality.

   a. If $N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT}{-\gamma\sigma\sqrt{T}}\right) < \alpha$ and $\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha) \leq 0$

      we get $\gamma > \frac{rT}{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha)}$.

   b. If $N\left(\frac{\gamma\left(\mu - \frac{1}{2}\sigma^2\right)T - rT}{-\gamma\sigma\sqrt{T}}\right) < \alpha$ and $\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha) > 0$

      we get $\gamma < \frac{rT}{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha)}$.

Second, the left inequality results in:

$$\frac{\left(\gamma\left(\mu+\frac{1}{2}(\gamma-1)\sigma^2\right)\right)\sqrt{T}}{-\gamma\sigma}-\frac{r\sqrt{T}}{-\gamma\sigma}+N^{-1}\left(1-\frac{1}{\delta}\right)\leq\frac{\gamma\left(\mu-\frac{1}{2}\sigma^2\right)T-rT}{-\gamma\sigma\sqrt{T}}$$

$$=\frac{\gamma\left(\mu-\frac{1}{2}\sigma^2\right)\sqrt{T}}{-\gamma\sigma}+\frac{-r\sqrt{T}}{-\gamma\sigma},$$

which implies:

$$\frac{2N^{-1}\left(1-\frac{1}{\delta}\right)}{\sigma\sqrt{T}}\leq\gamma.$$

In this case, the contract is a stop-loss policy with retention level b^* that solves

$N\left(\frac{\gamma\left(\mu-\frac{1}{2}\sigma^2\right)T-rT-\log\left(1-\frac{b^*}{p_0}\right)}{-\gamma\sigma\sqrt{T}}\right)=\alpha.$

If we solve for b^* we find

$$b^*=p_0\left(1-\exp\left(\left(\left(\mu-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}N^{-1}(\alpha)\right)\gamma-rT\right)\right)\tag{8}$$

3. If $N\left(\frac{\gamma\left(\mu-\frac{1}{2}\sigma^2\right)T-rT}{-\gamma\sigma\sqrt{T}}\right)<\alpha$ or $\frac{2N^{-1}\left(1-\frac{1}{\delta}\right)}{\sigma\sqrt{T}}>\gamma$.

In this case, the contract is a two-layer contract with lower and upper retention levels

a^*, b^*, where b^* is as in case 2 and a^* solves $N\left(\frac{\gamma\left(\mu-\frac{1}{2}\sigma^2\right)T-rT-\log\left(1-\frac{a^*}{p_0}\right)}{-\gamma\sigma\sqrt{T}}\right)=a$, which

similarly gives:

$$a^*=p_0\left(1-\exp\left(\gamma\left(\mu-\frac{1}{2}\sigma^2\right)T-rT+\gamma\sigma\sqrt{T}N^{-1}(a)\right)\right).$$

Note however that, $N^{-1}(a) = N^{-1}\left(N\left(\frac{|m|\sqrt{T}}{\eta} + N^{-1}\left(1-\frac{1}{\delta}\right)\right)\right) = \frac{|m|\sqrt{T}}{\eta} + N^{-1}\left(1-\frac{1}{\delta}\right)$.

Therefore,

$$a^* = p_0\left(1-\exp\left(\gamma\left(\mu-\frac{1}{2}\sigma^2\right)T - rT + \gamma\sigma\sqrt{T}\left(\frac{\left(\gamma\left(\mu+\frac{1}{2}(\gamma-1)\sigma^2\right)-r\right)\sqrt{T}}{-\gamma\sigma} + N^{-1}\left(1-\frac{1}{\delta}\right)\right)\right)\right)$$

$$= p_0\left(1-\exp\left(\gamma\left(\mu-\frac{1}{2}\sigma^2\right)T - rT \pm \gamma\left(\mu-\frac{1}{2}\sigma^2+\frac{1}{2}\gamma\sigma^2\right)T + rT + \gamma\sigma\sqrt{T}N^{-1}\left(1-\frac{1}{\delta}\right)\right)\right)$$

So finally, we get:

$$a^* = p_0\left(1-\exp\left(-\frac{1}{2}\gamma^2\sigma^2T + \gamma\sigma\sqrt{T}N^{-1}\left(1-\frac{1}{\delta}\right)\right)\right).$$

4.5 Calibration and simulation for VaR

Let us use the following calibration for the parameters: $= p_0 = 1, \mu = 0.01, r = 0.05, \alpha = 0.95, \delta = 1.1$, and let us consider $\gamma \in [-0.9, 0.1]$ and $\sigma \in [0.2, 0.9]$.

First, if we check $\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N^{-1}(\alpha) \geq 0$ then case 1 and case 2.a do not happen. From our observation in the parameter area that we study, this inequality holds; see the following **figure 6**. As a result, the lower retention level is always zero, i.e., $a^* = 0$.

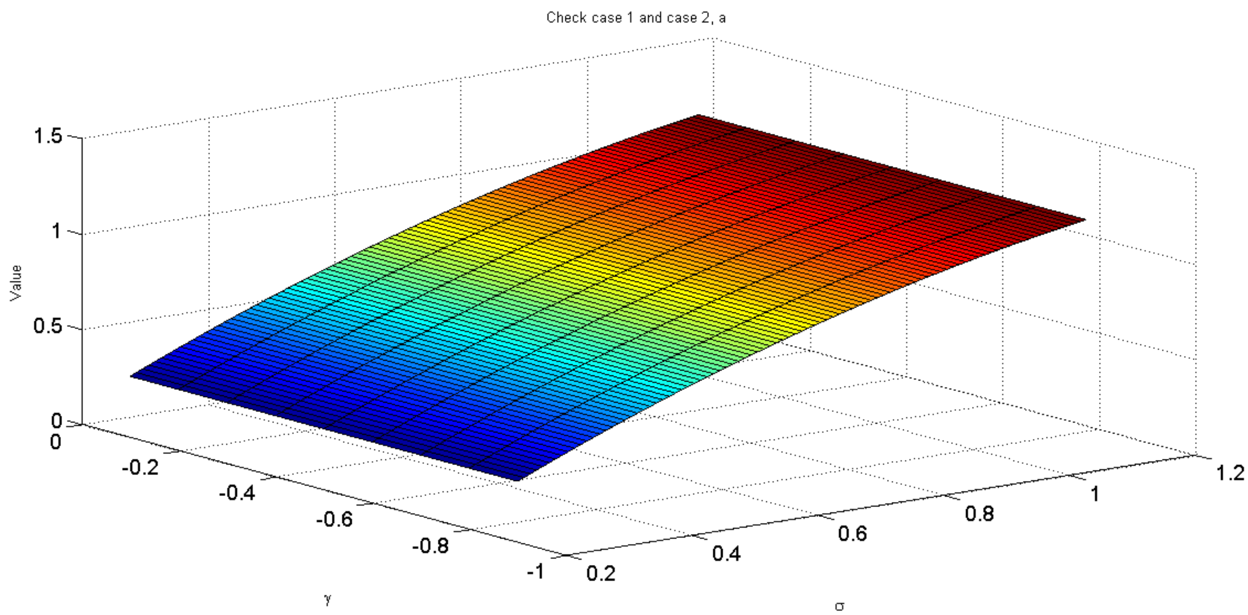

Figure 6: Verifying cases 1 and 2.a in the parameter space σ-γ

Second, we find the upper retention level b^* using equation (8) and the result is graphed in the following figure 7. As one can see, increases in σ or $|\gamma|$ increase the retention level b^*.

Third, as shown in figure 8, the contract prices will also increase with an increase in either σ or $|\gamma|$. This may appear counter-intuitive since for a constant σ, if $|\gamma|$ increases then ϕ gets closer to zero suggesting that a risk neutral producer is willing to pay more for a risk management tool designed in this framework. This effect warrants some explanation. In designing the contracts, the risk aversion parameter ϕ (or γ) does not only affect the price volatility (i.e., $-\gamma\sigma$), but also it will have an impact on the lower and the upper retention levels. This means that, even though a larger $|\gamma|$ will result in greater volatility, it also changes the optimal contract. Ultimately, the impact from changing a contract will dominate the volatility effect for a more risk-averse producer with larger ϕ and this results in a cheaper optimal contract.

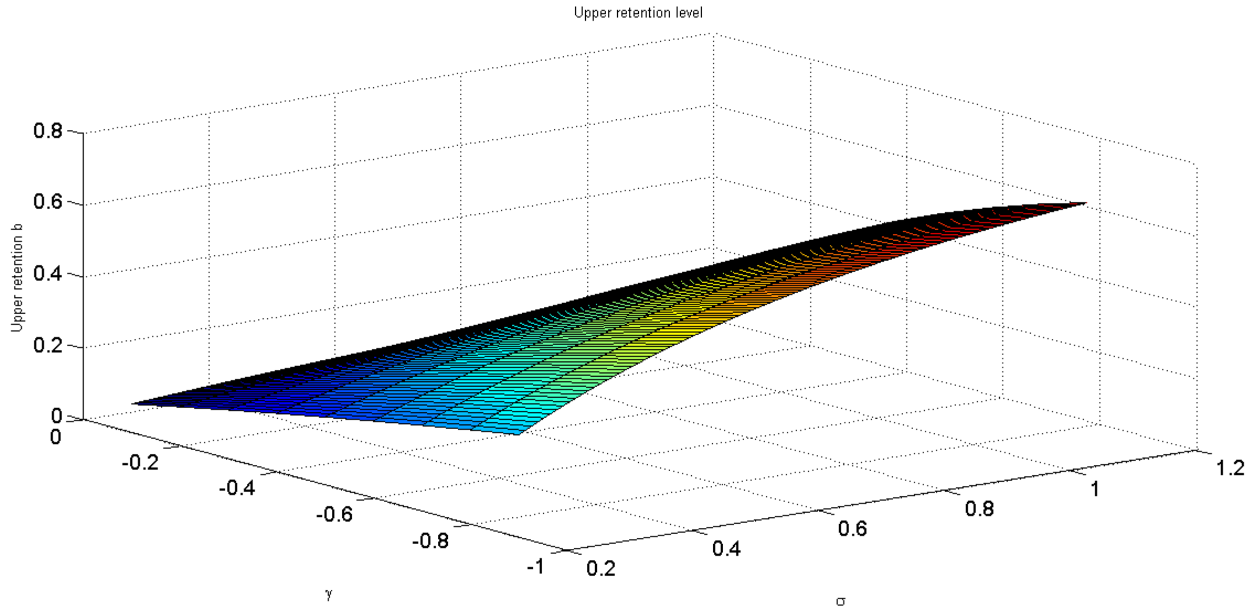

Figure 7: Upper retention level b* in terms of parameters σ-γ

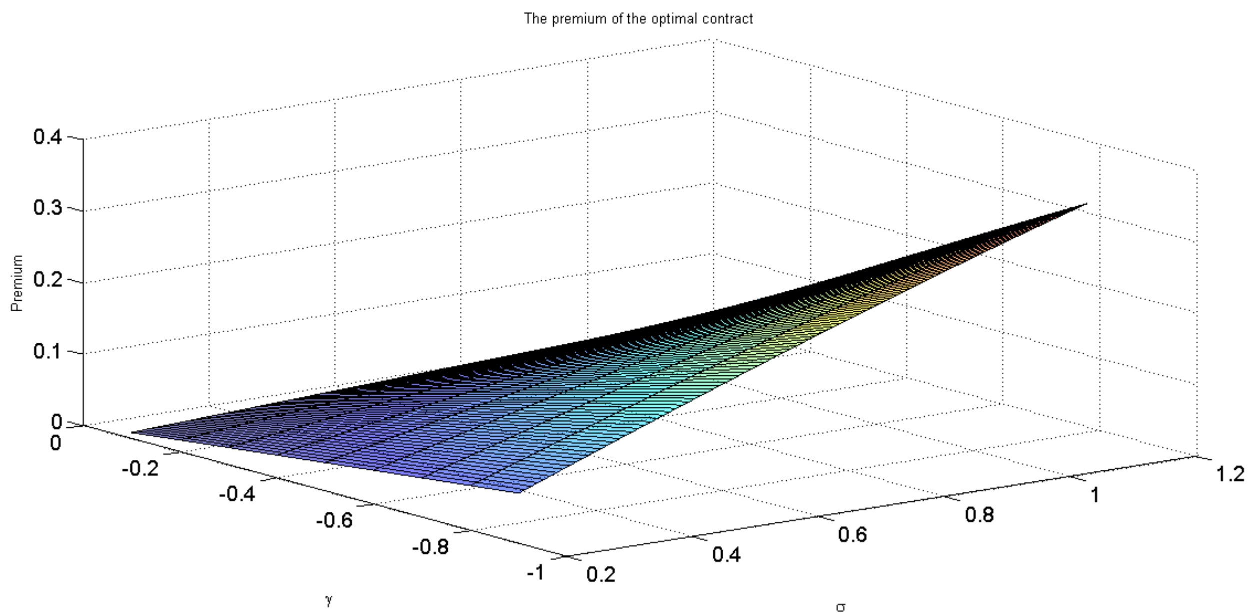

Figure 8: Premium of the optimal contract in σ-γ space

Further, for the premium we need to discuss what parameters can be admissible. That means determining which parameters can generate a nonnegative market price of risk. For that, we need to check for different parameters the nonnegativity of $(\nu - r)$, namely:

$$\gamma\left(\mu + \frac{1}{2}(\gamma - 1)\sigma^2\right) - r = \frac{1}{2}\gamma^2\sigma^2 + \gamma\left(\mu - \frac{1}{2}\sigma^2\right) - r \geq 0$$

Verifying this condition, we report in the following figure 9 a graph which represents the area for which the market price of risk is nonnegative.

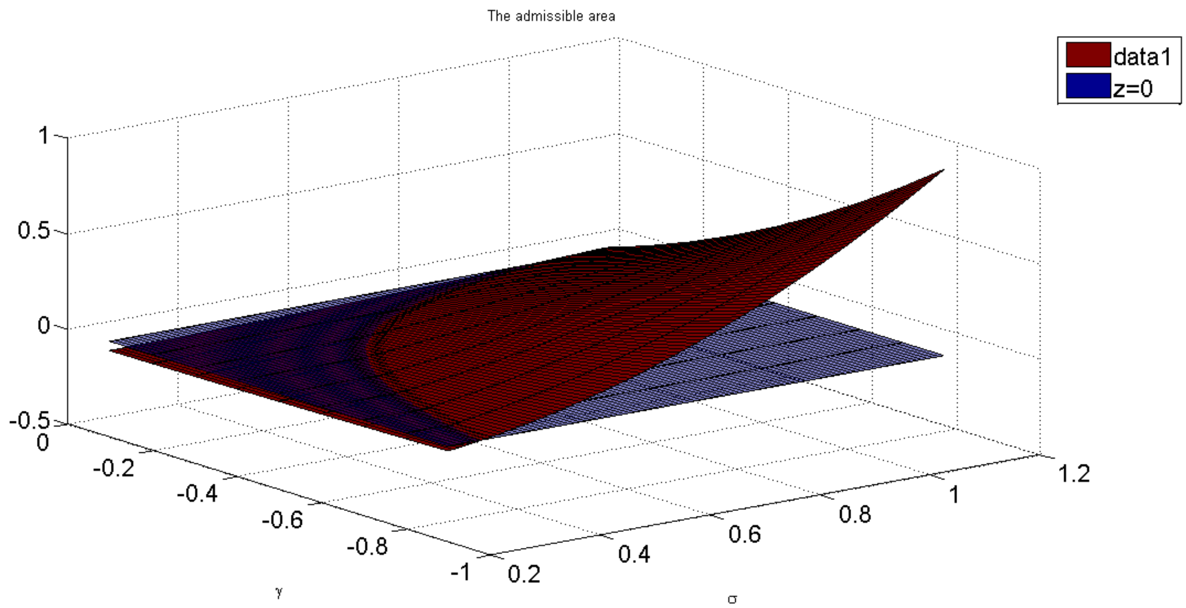

Figure 9: Admissible area to ensure a nonnegative market price of risk, σ-γ space

5. Empirical estimation of the model using commodity futures price data

5.1 Discretization of the process and estimation procedure

This section reports empirical estimates of the model parameters, obtained using a large dataset of commodity futures prices and a GMM estimation approach. Recall that we have shown that the price process has the following SDF:

$$\frac{dp_t}{p_t} = \nu dt + \eta dB_t, p_0 > 0$$

where $\nu = \gamma\left(\mu + \frac{1}{2}(\gamma - 1)\sigma^2\right)$ and $\eta = -\gamma\sigma$. Following standard procedures in the literature (e.g., Chan, Karolyi, Longstaff and Sanders [1992]), we estimate the parameters of this continuous-time model using a discrete-time econometric specification. We first need to derive the discrete-time version of the model, which is

$$\frac{\Delta p_t}{p_t} = \nu\Delta t + \eta\epsilon_{t+1}\sqrt{\Delta t}$$

Moreover, since $\Delta t = 1$,

$$\frac{p_{t+1} - p_t}{p_t} = \nu + \eta\epsilon_{t+1} \Longrightarrow p_{t+1} - p_T = \nu p_t + \eta\epsilon_{t+1}p_t$$

Then, considering $\epsilon_{t+1} = \eta\epsilon_{t+1}p_t$, we can write

$$\begin{gathered} p_{t+1} - p_t = \nu p_t + \epsilon_{t+1} \\ E[\epsilon_{t+1}] = 0, \qquad E[\epsilon_{t+1}^2] = \eta^2 p_t^2 \end{gathered} \tag{9}$$

using $E[\epsilon_{t+1}^2] = 1$. Now, define θ to be the vector of parameters with elements $(\gamma, \mu, \sigma)'$. Then, we let the vector $f_t(\theta)$ be:

$$f_t(\theta) = \begin{bmatrix} \epsilon_{t+1} \\ \epsilon_{t+1}p_t \\ \epsilon_{t+1}^2 - \eta^2 p_t^2 \\ (\epsilon_{t+1}^2 - \eta^2 p_t^2)p_t \end{bmatrix}$$

Under the null hypothesis implied by restrictions from eq above, $E[f_t(\theta)] = 0$. Then, the GMM procedure consists in replacing $E[f_t(\theta)]$ with $g_T(\theta)$, and using the T sample observations where

$$g_{T(\theta)} = T^{-1}\sum_{t=1}^{T} f_t(\theta)$$

and then choosing the parameters in θ that minimize

$$J_T(\theta) = g_T'(\theta)W_T(\theta)g_T(\theta) \tag{10}$$

where $W_T(\theta)$ is a positive-definite symmetric weighting matrix. To test the suitability of the model (i.e., the null hypothesis of correct model specification and the orthogonality conditions required for using GMM estimation), we compute Hansen's J-test (Sargan test of overidentification). A higher p-value for the J-test means that the instruments satisfy the orthogonality conditions and that the model specification is correct.

The GMM estimation method we use has two stages. First, we estimate the parameter values ν and η^2 that minimize eqn (10). Second, we replace the parameters in eqn (9) with the first-stage parameter estimates and rewrite it as follows:

$$p_{t+1} - p_t = \left(\gamma\mu + \frac{\hat{\eta}^2}{2} - \frac{\hat{\eta}}{2\gamma}\right) p_t + \epsilon_{t+1}$$

The resulting equation is then estimated by GMM to obtain the remaining parameters.

5.2 Description of the data and empirical results

The dataset consists of daily settlement prices for 19 commodity futures contracts obtained from Thomson Datastream, generally ending in late May 2018. At date *t* for a given commodity, there are a number M_t of contracts traded, where the first nearby is the nearest to maturity (expiry). Thus, the dataset is an unbalanced panel. For our purposes, only the nearby futures is needed. Rather than use the continuous series provided by Datastream for a given commodity futures contract (which introduces a splicing bias), we construct each time series of observations by rolling over from the first to second nearby contract on the 15th day of the month preceding maturity. This rollover method is standard and avoids including observations for dates near contract maturity, as the latter may not be reliable prices. All series use at least 10 years of daily observations. The specific length of the time series depends on data availability in Datastream.

Tables 1 and 2 present descriptive statistics for commodity futures prices and returns, respectively, for all contracts used in the analysis. Table 1 shows that most commodity price series are right-skewed and display negative excess kurtosis. Table 2 shows that for a majority of commodities, price log returns are right-skewed, while all display positive excess kurtosis.

Results for the estimated model, using GMM and applied to each of the 19 commodity series, are presented in table 3. First, for all 19 series, we fail to reject the null hypothesis of the Sargan J-test. Second, the results show that the main parameter of interest, γ, is around -1 for all series. For more than half of the commodities in our sample, the estimated γ is greater than 1 in absolute value. They are Chicago ethanol, Oman crude oil, cocoa, coffee, corn, soybean oil, oats, lean hogs, live cattle, feeder cattle, and gold. Meanwhile, the following commodities have an estimated γ less than 1 in absolute value: WTI crude oil, Dubai crude, wheat, sugar, rough rice, soybean meal, orange juice, and lumber. The parameter estimates suggest that prices of gold, ethanol, and soybean oil are the most sensitive to changing supply conditions, while wheat and soybean meal are the least sensitive. These new results can be used to improve risk management practice and derivative pricing, although such applications are beyond the scope of this paper.

6. Conclusion

The financialization of commodities has brought renewed interest in finance and risk management research to this asset class. Black's model [1976] remains the standard for pricing commodity derivatives, and most models are reduced-form in the style of Gibson and Schwartz [1990]. But to gain a deeper understanding of these markets, both for exchange-traded derivatives as well as insurance instruments, it is important to explicitly model the economic variables that determine the stochastic price process. To obtain explicit solutions to this problem, however, is not trivial.

Table 1: Descriptive statistics for daily settlement prices of 19 commodity futures contract time series.

Futures	Mean	Med.	Std	Q25	Q75	Skew	Exc. Kurt.	Nb.obs	Start	End
Energy										
Chicago ethanol	1.92	1.79	0.43	1.55	2.27	0.48	-1.02	2,984	12/15/06	5/23/18
WTI crude oil	76.46	84.24	23.68	54.44	96	0.02	-1.38	1,958	11/29/10	5/30/18
Dubai crude oil	63.36	61.23	39.71	40.27	103.7	-0.33	-1.3	2,608	1/2/08	12/29/17
Oman crude oil	79.67	79.1	25.12	55.62	104.3	0	-1.27	2,870	6/1/07	5/31/18
Storable agricultural										
Wheat	449.1	449.1	192.7	316.8	563.6	0.54	-0.18	6,428	10/4/99	5/23/18
Sugar	13.75	15.13	9.17	5.31	19.55	-0.06	-0.77	9,865	8/1/80	5/24/18
Cocoa	1937	1791	615.6	1,09	2192	0.85	-0.21	2,920	3/16/07	5/24/18
Coffee	145.71	136.5	43.03	120.1	162	1.2	1.5	3,819	10/6/03	5/24/18
Corn	468.55	421.1	115.6	384.5	550.4	1.03	0.19	2,622	8/8/08	5/25/18
Rough rice	1189	1192	306.6	994	1438	-0.11	-0.5	6,016	5/10/95	5/25/18
Soybean oil	41.51	38.81	9.68	33.32	49.84	0.7	-0.67	2,984	12/15/06	5/23/18
Oats	298	292	69.11	239.1	352	0.36	-0.65	2,878	5/15/07	5/23/18
Soybean meal	325.3	320	55.07	289.3	358.7	0.61	0.99	2,984	12/15/06	5/23/18
Orange juice	140.7	141	28.27	124.4	155.8	-0.02	0.38	2,822	8/1/07	5/24/18
Non-storable agricultural										
Lean hogs	78.23	79.25	14.9	68.16	86.68	0.54	1.21	2,679	2/18/08	5/24/18
Live cattle	118.5	119.4	21.23	102.7	132.7	0.15	-0.62	2,789	9/17/07	5/24/18
Feeder cattle	144.4	143.4	35.84	115.2	157.7	0.72	0.05	2,654	3/24/08	5/24/18
Other										
RL lumber	291.1	295	53.78	252.2	333.1	-0.31	-0.54	1,746	10/24/08	7/3/15
Gold	690.6	690.6	571.48	70.02	1106	0.38	-1.11	3,628	6/30/04	5/25/18

Notes: Time series are constructed from daily settlement price observations. The first nearby futures contract is used, except near maturity when the second nearby is used. The series is rolled over from the first to second nearby contract on the 15th day of the month preceding maturity, to avoid including observations near contract expiry. The statistics are, in order: mean, median, standard deviation, 25th quantile, 75th quantile, skewness, excess kurtosis, number of daily observations for the commodity, sample start date, sample end date.

Table 2: Descriptive statistics for daily price log returns of 19 commodity futures contract time series

Futures	Mean	Med.	Std	Q25	Q75	Skew	Exc. kurt.	Nb.obs	Start	End
Energy										
Chicago ethanol	0	0	0.02	-0.004	0.007	-3.33	33.56	2,983	12/15/06	5/23/18
Crude oil	0	0	0.02	-0.008	0.008	0.32	14.11	1,957	11/29/10	5/30/18
Dubai crude oil	-0.02	0	1.34	-0.01	0.009	-12.99	595.6	2,607	1/2/08	12/29/17
Oman crude oil	0	0	0.02	-0.009	0.009	0.12	5.71	2,869	6/1/07	5/31/18
Storable agricultural										
Wheat	0	0	0.07	-0.008	0.008	23	715.56	6,427	10/4/93	5/23/18
Sugar	0.02	0	1.47	-0.01	0.01	60.5	3,660.3	9,864	8/1/80	5/24/18
Cocoa	0	0	0.03	-0.008	0.008	4.79	131.24	2,919	3/16/07	5/24/18
Coffee	0	0	0.02	-0.01	0.01	1.64	24.99	3,818	10/6/03	5/24/18
Corn	0	0	0.02	-0.009	0.008	0.5	9.01	2,621	8/8/08	5/25/18
Rough rice	0	0	0.02	-0.006	0.006	25.48	1,150.4	6,015	5/10/95	5/25/18
Soybean oil	0	0	0.01	-0.008	0.008	0.22	2.74	2,983	12/15/06	5/23/18
Soybean meal	0	0	0.02	-0.008	0.008	-1.5	20.24	2,983	12/15/06	5/23/18
Orange juice	0	0	0.02	-0.01	0.01	0.27	3.88	2,821	8/1/07	5/24/18
Oats	0	0	0.02	-0.009	0.009	0.08	11.43	2,877	5/15/07	5/23/18
Non-storable agricultural										
Lean hogs	0	0	0.02	-0.008	0.008	1.58	36.13	2,678	2/18/08	5/24/18
Live cattle	0	0	0.01	-0.004	0.005	-1.96	37.1	2,788	9/17/07	5/24/18
Feeder cattle	0	0	0.01	-0.005	0.005	-0.25	3.36	2,653	3/24/08	5/24/18
Other										
RL lumber	0	0	0.02	-0.008	0.006	4.36	74.26	1,745	10/24/08	7/3/15
Gold	0	0	0.02	-0.006	0.008	-21.06	899.3	3,627	6/30/04	5/25/18

Notes: Time series are constructed from daily settlement price observations. The first nearby futures contract is used, except near maturity when the second nearby is used. The series is rolled over from the first to second nearby contract on the 15th day of the month preceding maturity, to avoid including observations near contract expiry. The statistics are, in order: mean, median, standard deviation, 25th quantile, 75th quantile, skewness, excess kurtosis, number of daily observations for the commodity, sample start date, sample end date.

Table 3: Estimates of the model parameters for 19 commodity futures, using GMM

	First-stage			Second-stage				N.obs.
Futures contract	**ν (s.e.)**	**η (s.e.)**	**J-test (p-value)**	**γ (s.e.)**	**μ (s.e.)**	**σ**	**J-test (p-value)**	
Energy								
Chicago ethanol	.0002	.0173**	4.515	-1.368**	.0007*	.0141	1.347	2984
	(.0004)	(.0063)	(.105)	(.06)	(.0003)		(.51)	
WTI crude oil	-.0002	.0173**	2.551	-.94**	.0005	.0173	2.551	1958
	(.0003)	(.0044)	(.279)	(.041)	(.0003)		(.279)	
Dubai crude	-.0002	.0141**	1.36	-.977**	.0002	.0141	1.426	2608
	(.0003)	(.001)	(.507)	(.033)	(.0003)		(.49)	
Oman crude	.001	.0173**	2.348	-1.152**	.0002	.0173	5.416	2870
	(.0003)	(.0045)	(.309)	(.03)	(.0003)		(.07)	
Storable agricultural								
Wheat	.007**	.0447**	.645	-.846**	.0029**	.0548	3.971	6428
	(.0005)	(.02)	(.71)	(.105)	(.0006)		(.137)	
Sugar	-.0008	.0548**	1.969	-.955**	.001*	.0265	1.728	9865
	(.0004)	(.0071)	(.374)	(.043)	(.0005)		(.422)	
Cocoa	.0008°	.0316**	.013	-1.202**	.0224	.0265	3.983	2920
	(.0005)	(.0141)	(.993)	(.082)	(.02)		(.137)	
Coffee	.0002	.02**	3.308	-1.007**	.00003	.02	4.199	3819
	(.0004)	(.0045)	(.191)	(.025)	(.096)		(.123)	
Corn	.0001	.0173**	2.944	-1.013**	.0002	.0173	2.91	2622
	(.0003)	(.0044)	(.229)	(.0003)	(.0003)		(.233)	
Rough rice	.00001	.0141**	2.135	-.89**	.0003	.0173	2.329	6016
	(.0002)	(.0032)	(.344)	(.026)	(.0002)		(.312)	
Soybean meal	.025**	.0316**	3.475	-.521**	.002**	.0632	5.642	2984
	(.0003)	(.0045)	(.176)	(.014)	(.0006)		(.06)	
Soybean oil	.0004	.0141**	3.702	-1.665**	.0005*	.00836	1.328	2984
	(.0003)	(.0032)	(.157)	(.074)	(.0002)		(.515)	
FC orange juice	.007**	.0224**	1.775	-.875**	.0003	.0264	1.198	2822
	(.0004)	(.0045)	(.412)	(.024)	(.0005)		(.549)	
Oats	.0001	.0173**	3.429	-1.06**	.0002	.0173	3.324	2878
	(.0004)	(.0055)	(.181)	(.034)	(.0003)		(.19)	
Nonstorable agricultural								
Lean hogs	-.0004	.0173**	2.68	-1.13**	.0003	.0141	2.661	2679
	(.0003)	(.0055)	(.262)	(.05)	(.0003)		(.264)	
Live cattle	.0006**	.01**	1.331	-1.064**	-.00001	.01	2.745	2789
	(.0002)	(.0032)	(.514)	(.042)	(.0002)		(.254)	
Feeder cattle	.0001	.01**	2.15	-1.044**	-.00001	.01	2.15	2654
	(.0002)	(.00224)	(.341)	(.026)	(.0002)		(.341)	
Other								
RL lumber	-.00004	.0141**	4.532	-.989**	.0002	.0141	4.532	1746
	(.0004)	(.00316)	(.104)	(.031)	(.0004)		(.104)	
Gold	.0004	.0141**	3.35	-1.35**	-.0001	.01	3.111	3628
	(.0003)	(.0089)	(.187)	(.19)	(.0002)		(.211)	

Notes: This table presents estimates of the parameters of the price process described in this paper, for a range of different futures contracts. The data are obtained from Thomson Datastream. Prices are daily settlement. The following discrete-time equation is estimated using the Generalized Method of Moments (see the paper for the derivation of this equation, and see the algorithm described in appendix A). The null hypothesis of the Hansen J-Test is that the overidentification restrictions are valid, so the instruments are valid. Statistical significance is denoted using ** (1% level), * (5% level), and ° (10% level).

$p_{t+1} - p_t = \nu p_t + \epsilon_{t+1}, \quad E[\epsilon_{t+1}] = 0, \quad E[\epsilon_{t+1}^2] = \eta_t^2 p_t^2, \ \nu = \gamma\left(\mu + \frac{1}{2}(\gamma - 1)\sigma^2\right)$, and $\eta = -\gamma\sigma$

This paper develops a more structural framework to price commodity derivatives and optimal insurance contracts. The contingent claim methodology that is proposed here is based on standard risk-neutral valuation arguments, but inspired by the rational storage models of Deaton-Larocque [1992, 1995, 1996]. It can be then applied to exchange-traded or OTC derivatives, or to insurance instruments.

In this paper, we show how to obtain commodity-specific pricing solutions in terms of deeper economic parameters such as the price elasticity of demand for a given commodity, as well as the loss function that best describes the trader or hedger (e.g. Value-at-risk, or conditional Value-at-risk). We also consider the role played by the risk specification among a class of distortion risk measures. Results are presented for some risk management applications, where optimal contract types are obtained in terms of the parameter space. In some cases, no contract is optimal. These findings should be useful for academics and practitioners in commodity finance, derivatives, risk management and insurance.

The analysis described in the paper also suggests some potentially useful avenues for further research. One is to take the model to data on commodity futures and options contracts to recover estimates of the parameters, and compare pricing accuracy relative to commonly-used methods. A second would be to investigate empirically the no-optimal-contract case by relating the model's prediction to data on contract trading volume and interest.

7. Acknowledgments

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