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Optimal Closing Benchmarks

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Optimal Closing Benchmarks

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Abstract

In financial markets, the closing price serves as an important benchmark. We introduce a market model to analyze the stability of the closing price with presence of three types of volume: distorting volume, volume that targets the closing price, and volume that is unrelated to the closing price. The optimal closing price is either the price from an auction or the volume weighted average price (VWAP) from regular trading only, explaining the prevalence of these closing benchmarks on financial markets. A succinct condition depending on the different volume types indicates when the inclusion of a closing auction is optimal.

Keywords: benchmark stability, closing auction, volume weighted average price, VWAP
JEL Codes: G14, G18, D44, D82

1 Introduction

Closing prices are important benchmarks in financial markets since many investors use them as reference points. In practice, two methods to determine the closing price are predominant: a closing auction, used for stocks at all developed markets and some emerging markets, and the volume weighted average price (VWAP), common in futures markets and several emerging stock markets; see FTSE Russell [2019]. A primary concern with benchmark design is robustness to manipulating or distorting orders. For example, Aspris et al. [2020] analyze how changes on benchmarks in precious metals markets could reduce manipulation. Another example is the move of the Paris Bourse (currently Euronext Paris) to a closing auction, whose main reason was that some relatively small orders could have changed closing prices, as Pagano and Schwartz [2003] note. The stability issue of closing prices is underlined by judicial and empirical evidence for their manipulation [Commodity Futures Trading Commission, 2013, Hillion and Suominen, 2004].

The closing price of a stock is especially important to mutual funds as purchases and redemptions of their shares occur at the day's closing price of the funds' constituent assets. This creates the desire for mutual funds to buy and sell stocks at their closing prices. As a result, there is a significant portion of volume that follows prices used to create the benchmarks. This is especially noticeable in markets after the introduction of a closing auction.[1] Kandel et al. [2012] as well as Hagströmer and Nordén [2014] find that trader patience increases after the introduction of a closing auction as a result of improved liquidity during the end of regular trading. It would then be prudent to consider the effect of a benchmark change that results from such benchmark targeting volume.

We include this volume as a feature of our market model, which also has linear price impact during regular trading and distorting volume. We take the perspective of a benchmark administrator, who wishes to minimize the impact of distorting volume on the benchmark. The administrator decides (1) whether or not to include an auction in the market, (2) the benchmark weight of the auction if it is included, and (3) the distribution of the remaining weights in the regular trading. We find a unique optimal benchmark that minimizes the impact of distorting volume under worst-case scenario assumptions, as well as a succinct condition for when the inclusion of a closing auction in this market is optimal. We also find that in the exclusion of a closing auction, the unique optimal benchmark is VWAP and give a method to find the optimal start time of the window, over which VWAP is computed.

On the one hand, our result explains the prevalence of VWAP and auction benchmarks in practice; on the other hand, it gives a criterion when VWAP or an auction is preferable, even determining the optimal window length in the case of VWAP. This condition can be applied to a market with an estimation of benchmark targeting volume relative to volume that is unrelated to the benchmark and provides insight into why these two kinds of benchmarks appear in different markets.

VWAP-type benchmarks have been found to be optimal in different settings and questions. For example, Duffie and Dworczak [2018] examine optimal benchmarks in settings where data from transactions or reports of agents whose profits depend on such benchmarks are available. They find a benchmark that puts small weight on small transactions and nearly

[1]For large developed markets of stocks, closing auctions account for as much as 20% of the daily traded volume [Golden, 2018].

equal weight on all large transactions. When such data is unavailable, VWAP emerges as the optimal benchmark. Under certain conditions, Baldauf et al. [2020] find VWAP as the unique optimal benchmark for the principal agent problem of a client contracting a broker to purchase a large amount of shares. While a closing auction based benchmark may seem more attractive to agents in the marketplace who target the benchmark, execution algorithms with VWAP benchmarks are readily available and well studied [Cartea and Jaimungal, 2016, Frei and Westray, 2015, Guéant and Royer, 2014, Humphery-Jenner, 2011, Kato, 2015].

A VWAP benchmark includes volume that is unrelated to the specific benchmark, providing additional stability. However, closing auctions can help absorb liquidity shocks by pooling volume. Foucault et al. [2005] and Roşu [2009] study dynamic models of limit order markets with strategic liquidity traders of varying patience. Their conclusions indicate that the introduction of a closing auction can increase trader patience and thus reduce spreads. These theoretical findings are confirmed by Pagano and Schwartz [2003] and Kandel et al. [2012] for the stock markets of the Borsa Italiana and the Paris Bourse, where they document a significant reduction in spread and volatility around the close when closing auctions were introduced. For NASDAQ-OMXS30 index futures, which have a closing auction as an exception among futures markets, Hagströmer and Nordén [2014] find that the introduction of a closing auction improved closing price accuracy.

The remainder of this paper is organized as follows. Section 2 describes a market model with linear price impact during regular trading and three types of volume: benchmark targeting volume, distorting volume, and noise volume that is unrelated to the benchmark. We use the hypothetical shape of an order book under linear price impact to build supply and demand curves present in an auction and derive the price impact of an order submitted to the auction. Section 3 introduces the optimization that the benchmark administrator faces. We state and discuss the main result, that is, the unique optimal benchmark and the aforementioned condition for auction inclusion. Its proof is contained in Appendix A.

2 The market model

We consider a discrete market model with T periods consisting of $T-1$ periods of regular trading and possibly an auction at time T.[2] We will later introduce a benchmark administrator, who decides about the closing benchmark and whether or not an auction takes place. The market is made up of three types of volume:

- Noise or outside volume, denoted as u_i, which does not follow the benchmark. This

[2] Alternatively, we could consider a continuous-time model for regular trading. Our results continue to hold in such a continuous-time version under a specification analogous to the discrete model that we present.

volume is unsigned and we assume it to be strictly positive in any regular trading period so that $u_i > 0$ almost surely for every $i = 1, \ldots, T-1$.

- Benchmark targeting volume, denoted as α_i, which is also unsigned ($\alpha_i \geq 0$).
- Distorting volume, denoted as v_i and summing to V; this volume is made up of potential distortions to the benchmark. We assume that all v_i have the same sign. This assumption will allow us to analyze the maximal price impact across the different trading periods in terms of the total distorting volume V. The assumption corresponds to a worst-case scenario in that the distortions in all periods go in the same direction.

All of u_i, α_i, v_i, and V are random variables. The prices observed are modelled as

$$p_i = \tilde{p}_i + c\frac{v_i}{u_i + \alpha_i} \quad \text{for} \quad i = 1, \ldots, T-1,$$

where we do not make any assumptions on the underlying price process $\tilde{p}_i$, which can have arbitrary distribution and dependence structure. Such a model of temporary price impact is well founded in the literature. Indeed, linear price impact (in v_i) is consistent with price impact models based on adverse selection [Kyle et al., 2018]. The ratio $\frac{v_i}{u_i+\alpha_i}$ makes the price impact dimensionless [Almgren et al., 2005]. The linear price impact model in regular trading can be seen as each period having an order book with a constant amount of sell orders above $\tilde{p}_i$ and a symmetric amount of buy orders below of $\tilde{p}_i$. Indeed, the price changes by one unit for each volume of $\frac{u_i+\alpha_i}{c}$. A continuous version of this order book is shown below in Figure 1.

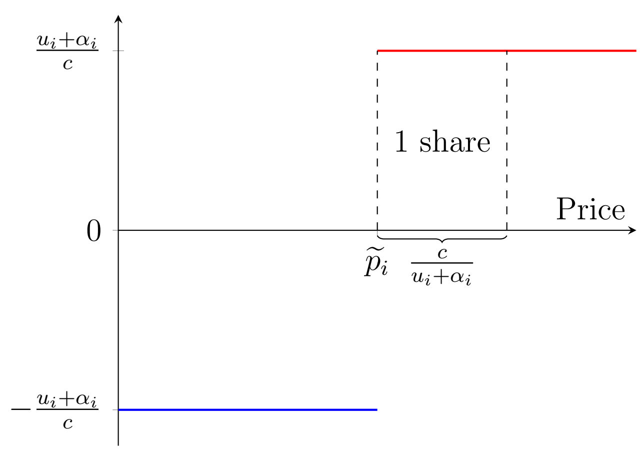

Figure 1: the order book such that buying v_i shares increases the price by $c\frac{v_i}{u_i+\alpha_i}$

In the auction, when all volume is pooled, we can represent the price as the intersection of supply and demand curves. We assume that volume unrelated to the auction, u_T, is zero.

The supply and demand curves correspond to the aggregation of the sell and buy orders, respectively. Because our model is based on constant amounts of sell and buy orders at different prices, this aggregation leads to supply and demand curves that are linear and have the same slope in absolute value, as the integration of a constant results in a linear function. A distorting buy order will shift the demand curve to the right while a distorting sell order will shift the supply curve to the right; compare Figure 2.

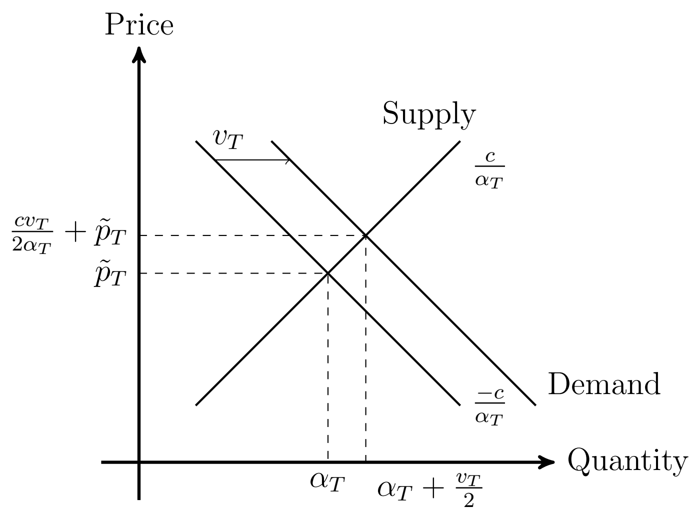

Figure 2: the shifting of the demand curve in the auction from a distorting buy order v_T

Since the slopes of the supply and demand lines are the negative of one another, shifting the demand curve by v_T causes the new equilibrium quantity to shift by only $v_T/2$, which results in a price impact of $c\frac{v_T}{2\alpha_T}$. Consequently, we model the auction price as $p_T = \tilde{p}_T + c\frac{v_T}{2\alpha_T}$, provided that the benchmark administrator decides to have an auction.

3 The benchmark administrator's optimization

The administrator desires to choose a benchmark such that the impact of the distorting orders, v_i, on the benchmark is minimized. The administrator chooses a nonnegative weight β_i for trading period i, where we normalize $\sum_{i=1}^{T} \beta_i = 1$. Since an auction consists only of volume related to the benchmark, choosing $\beta_T = 0$ is equivalent to having no auction. The impact of the distorting orders on the benchmark is given by $\sum_{i=1}^{T} \beta_i(p_i - \tilde{p}_i)$, which depends on the benchmark weights β_i, both directly as the weights of the prices and indirectly through the benchmark targeting volume that is reflected in the prices p_i.

Assumption. *We replace α_i by $A\beta_i$ for all $i = 1, \ldots, T$ and a random variable A, which is strictly positive almost surely.*

This assumption means that we consider volume that perfectly matches the benchmark and thus is proportional to its weights, rather than traders targeting the benchmark with uncertainty and matching error. Of course, this assumption is idealistic. Nonetheless, it is not too far away from the reality when benchmark weights depend on the relative volume curve. Indeed, the relative volume curve can be forecast to a fair degree of accuracy (e.g., Exhibit 9 of Satish et al. [2014]). We also note that $\beta_T > 0$ whenever the auction exists so that the auction price $p_T = \tilde{p}_T + c\frac{v_T}{2A\beta_T}$ is then well defined.

Because the benchmark is used at the close, the administrator is also worried about the cost that comes from starting early with nonzero benchmark weights. This cost can be related to the need of a longer time interval for computing the benchmark, which leads to additional complexity and means clients of mutual funds would not be able to submit orders during this time. Let M be the first nonzero weight of the benchmark so that $\beta_M \neq 0$ and $\beta_i = 0$ for all $i < M$. We model the costs of starting the benchmark computation early by $Q(M)$, where Q is a strictly decreasing function.

To see why it is desirable to have the benchmark start later in the day, consider a mutual fund whose inflows occur at the benchmark and a benchmark that starts in the last 10 periods of trading. If an underlying asset in the mutual fund's price takes a significant jump with 5 periods left in the day, one could purchase the mutual fund and obtain the constituent asset at a price that is based on the last 10 periods and likely to be less than the closing price. To combat this, mutual funds typically do not allow trades during the benchmark building period.

We assume that the administrator chooses the benchmark at the end of the period and can observe prices, the outside volumes u_i, and the total benchmark targeting volume A. We denote by $\mathcal{F}$ the σ-algebra generated by $p_1, \ldots, p_T$, $u_1, \ldots, u_T$, and A so that the benchmark administrator will make $\mathcal{F}$-measurable choices. Thus, the benchmark administrator faces the optimization

$$\min_{\beta_i, M} \left\{ \max_{v_i} E\left[\left| \sum_{i=M}^{T} \beta_i (p_i - \tilde{p}_i) \right| + Q(M) \right] \right\}, \tag{1}$$

where the maximization is over all random variables v_i such that $\sum_{i=1}^{T} v_i = V$ with all v_i having the same sign as V; the minimization is over all β_i and M that are $\mathcal{F}$-measurable and satisfy $\sum_{i=M}^{T} \beta_i = 1$ with $\beta_i = 0$ for $i < M$. Our result remains unchanged if the expectation in (1) is replaced by the $\mathcal{F}$-conditional expectation. We denote by M^* an $\mathcal{F}$-measurable random variable that minimizes

$$\frac{cE\left[|V| | \mathcal{F}\right]}{A + \max\{\sum_{j=M}^{T-1} u_j, A\}} + Q(M) \tag{2}$$

over M in $\{1, 2, \ldots, T\}$. Because A and all u_j are $\mathcal{F}$-measurable, the random variables in (2) for every fixed M are $\mathcal{F}$-measurable. Therefore, also the minimum over these T random variables is $\mathcal{F}$-measurable, and there exists an $\mathcal{F}$-measurable minimizer M^*.

Main Result. *There exists a unique optimal benchmark.*

On $M^* = T$, there is a closing auction and the optimal benchmark puts all weight into the closing auction with $\beta_T^* = 1$ and $\beta_i^* = 0$ for all $i < T$.

On $M^* < T$, there is no closing auction and the optimal benchmark is VWAP with $\beta_j^* = \frac{A\beta_j^* + u_j}{A + \sum_{\ell=M^*}^{T-1} u_\ell} = \frac{u_j}{\sum_{\ell=M^*}^{T-1} u_\ell}$ for $j = M^*, \ldots, T-1$ and $\beta_T^* = 0$ and $\beta_i^* = 0$ for $i < M^*$.

Depending on the realization of M^*, the decision of whether or not to have an auction is a simple yes or no. This is consistent with what we see in practice: VWAP and auction closing prices are predominant, but no market computes the closing price as a weighted combination of VWAP and the auction price. The decision between VWAP and closing auction as well as when to start the VWAP window can be tweaked depending on a benchmark administrator's perceived penalty for starting the benchmark period early through the choice of the penalty $Q(M)$, which has no other effect on the structure of the optimal benchmark.

In the case $\sum_{j=1}^{T-1} u_j \leq A$ almost surely, meaning that less volume is unrelated to the benchmark than volume targeting it, we can see that the ratio $\frac{cE[|V||\mathcal{F}]}{A + \max\{\sum_{j=M}^{T-1} u_j, A\}}$ in (2) does not depend on M. Because Q is strictly decreasing, it is then optimal to choose $M^* = T$, and we can deduce from our main result the following corollary.

Corollary. *If $\sum_{j=1}^{T-1} u_j \leq A$ almost surely, there is a closing auction and the optimal benchmark puts all weight into the closing auction with $\beta_T^* = 1$ and $\beta_i^* = 0$ for all $i < T$.*

By contrast, if the distributions of $\sum_{j=1}^{T-1} u_j$ and A overlap or even if $\sum_{j=1}^{T-1} u_j > A$ almost surely, then one must choose $Q(M)$ explicitly and optimize (2) over M^* either analytically or numerically to determine if a closing auction is optimal or not. If $\sum_{j=1}^{T-1} u_j > A$ almost surely and Q has a small slope in absolute value such that its marginal impact on (2) is minimal, then VWAP will be the optimal benchmark.

Our result suggests the following procedure that assists in the optimal benchmark choice:

1. Determine a maximal acceptable time window for a closing price mechanism, which will depend on the particular market.

2. Quantify the total volume A that follows the benchmark and the outside volume that is unrelated to the benchmark. This distinction in trading volumes can be achieved by comparing historical volume in the closing period with volumes in comparable periods during the day.

3a. If A exceeds the total outside volume over the acceptable window, then a closing auction is optimal.

3b. If A is less than the total outside volume over the acceptable window, we introduce a function Q that models the cost of starting the benchmark computation early, determine the minimizer M^* of (2), and apply our main result.

As an example, we report in Table 1 (on the next page) a volume comparison for the 30 stocks in the Dow Jones Industrial Average (DJIA) during April 2020. For these stocks, the closing price is determined in auctions. We observe that the auction volume exceeds the traded volume during the 15 minutes and 30 minutes before the close for most of these stocks, with the exceptions of Apple (AAPL), Boeing (BA), and Walt Disney Co. (DIS). Following the procedure described above, this implies by step 3a that a closing auction is indeed optimal in most of the stocks in the DJIA if the acceptable window is 30 or less minutes. If an optimal closing mechanism were chosen individually for AAPL, BA, and DIS with an acceptable window of 30 minutes, the above step 3b shows that further analysis and the modeling of the function Q depending on the benchmark administrator's preferences would be necessary.

Generally, the bigger the benchmark targeting volume is relative to the volume that is unrelated to the closing price, the more likely the minimizer of (2) will be T so that a closing auction is optimal. Interestingly, we observe lower volume around the close in emerging stock markets compared to developed stock markets, which saw a rise in volume around the close [Golden, 2018]. The above example and this observation motivate why developed stock markets moved to closing auctions over the last two decades while many emerging stock markets still use a VWAP benchmark.

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	benchmark volume	15 min before the close		30 min before the close	
AAPL	101.5	71.4	-29.7%	106.0	4.4%
AXP	24.6	17.1	-30.5%	24.5	-0.3%
BA	16.3	40.0	145.0%	67.6	313.9%
CAT	22.6	12.4	-45.2%	16.9	-25.3%
CSCO	134.9	54.5	-59.6%	74.4	-44.9%
CVX	58.1	29.4	-49.3%	41.7	-28.3%
DIS	41.5	28.9	-30.3%	43.3	4.4%
DOW	33.1	16.2	-51.0%	22.5	-32.1%
GS	16.6	11.0	-33.8%	15.6	-6.0%
HD	26.6	12.2	-54.1%	16.9	-36.4%
IBM	31.3	14.1	-55.0%	19.5	-37.9%
INTC	175.1	67.2	-61.6%	90.9	-48.1%
JNJ	67.4	23.9	-64.5%	33.6	-50.1%
JPM	99.4	46.3	-53.4%	66.9	-32.7%
KO	100.7	44.4	-55.9%	60.7	-39.7%
MCD	24.2	11.1	-54.0%	15.5	-36.0%
MMM	23.7	9.4	-60.3%	12.6	-46.8%
MRK	69.9	26.2	-62.5%	36.7	-47.5%
MSFT	179.8	90.2	-49.9%	126.6	-29.6%
NKE	39.8	17.5	-56.1%	24.3	-38.9%
PFE	137.7	55.6	-59.6%	77.6	-43.6%
PG	72.4	30.6	-57.7%	45.1	-37.7%
TRV	14.8	6.2	-57.7%	8.1	-45.4%
UNH	28.5	13.1	-54.1%	17.8	-37.6%
UTX	7.2	4.6	-35.3%	6.7	-6.3%
V	74.1	24.2	-67.3%	34.2	-53.8%
VZ	92.6	38.8	-58.1%	53.9	-41.8%
WBA	39.7	16.9	-57.3%	22.2	-44.1%
WMT	42.3	19.4	-54.3%	28.0	-34.0%
XOM	109.5	62.2	-43.2%	89.4	-18.4%
DJIA	**1,905.7**	**915.1**	**-52.0%**	**1,299.4**	**-31.8%**

Table 1: Volume comparison of the 30 stocks in the Dow Jones Industrial Average (DJIA) during April 2020. All volumes are in million shares, and percentages are relative to the benchmark volume. The benchmark volume corresponds to volume in the closing auction at 4 pm, which is compared to the volumes between 3:45 pm and 4 pm (15 min before the close) and between 3:30 pm and 4 pm (30 min before the close). For all stocks except for BA, the benchmark volume surpasses the volume 15 minutes before the close. The comparison for the 30-minute window is similar, but there are now three stocks (AAPL, BA, and DIS) with more traded volume before the close than in the auction. While the volume comparison for BA stands out, also in other periods its 15-minute and 30-minute volumes before the close clearly, albeit less extremely exceed its benchmark volume (e.g., in April 2019, 15-minute and 30-minute volumes for BA were 28.9% and 87.2% higher than the benchmark volume).

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A Proof of the main result

Step 1: inner maximization problem
We first write the inner maximization problem in (1) as

$$\max_{v_i} E\left[\left|\sum_{i=M}^{T} \beta_i(p_i - \tilde{p}_i)\right|\right] = c \max_{v_i} E\left[\left|\sum_{i=M}^{T-1} \frac{\beta_i v_i}{u_i + A\beta_i} + \frac{v_T}{2A}\mathbb{1}_{\beta_T \neq 0}\right|\right]. \quad (3)$$

We will use the auxiliary result that for any nonnegative $\kappa_M, \ldots, \kappa_T$, we have

$$\max_{v_i} \left|\sum_{i=M}^{T} \kappa_i v_i\right| = |V| \max_{j=M,\ldots,T} \kappa_j. \quad (4)$$

This result follows from

$$\left|\sum_{i=M}^{T} \kappa_i v_i\right| \leq \left|\max_{j=M,\ldots,T} \kappa_j \sum_{i=M}^{T} v_i\right| = |V| \max_{j=M,\ldots,T} \kappa_j,$$

where we used for the inequality that all v_i have the same sign. Equality holds if $v_{i^*} = V$ for $i^* = \text{argmax}_{i=M,\dots,T}\kappa_i$ (if there are several i with maximal κ_i, we can choose i^* arbitrarily among them) and $v_i = 0$ for all $i \neq i^*$.

Using that the worst case of an expectation is given by the worst case in each scenario, we apply (4) to (3) in each scenario with $\kappa_i = \frac{\beta_i}{u_i + A\beta_i}$ for $i = M, \dots, T-1$ and $\kappa_T = \frac{1}{2A}\mathbb{1}_{\beta_T \neq 0}$, which yields

$$\max_{v_i} E\left[\left|\sum_{i=M}^{T} \beta_i(p_i - \tilde{p}_i)\right|\right] = cE\left[|V| \max\left\{\max_{j=M,\dots,T-1} \frac{\beta_j}{u_j + \alpha_j}, \frac{1}{2A}\mathbb{1}_{\beta_T \neq 0}\right\}\right]. \tag{5}$$

Step 2: outer minimization problem

Thanks to (5), the optimization problem (1) becomes

$$\min_{\beta_i, M}\left\{E\left[c|V| \max\left\{\max_{j=M,\dots,T-1} \frac{\beta_j}{u_j + A\beta_j}, \frac{1}{2A}\mathbb{1}_{\beta_T \neq 0}\right\} + Q(M)\right]\right\}. \tag{6}$$

Step 2a: outer minimization problem over $\beta_M, \dots, \beta_{T-1}$

For given $M < T$ and $\beta_T < 1$, we first analyze the minimization of

$$\max_{j=M,\dots,T-1} \frac{\beta_j}{u_j + A\beta_j}$$

over $\beta_M, \dots, \beta_{T-1}$. We note that $\frac{\beta_j}{u_j + A\beta_j} = \frac{1}{A}\big(1 - \frac{u_j}{u_j + A\beta_j}\big)$ is increasing in β_j. In the minimum over $\beta_M, \dots, \beta_{T-1}$, we need equality of the ratios

$$\frac{\beta_i}{u_i + A\beta_i} = \frac{\beta_j}{u_j + A\beta_j} \quad \text{for all} \quad i, j = M, \dots, T-1,$$

as otherwise, we could lower $\max_j \frac{\beta_j}{u_j + A\beta_j}$ by making some β_j smaller and adding weight to another β_i. This leads to the optimal β_j^* of the form

$$\beta_j^* = a(u_j + A\beta_j^*),$$

where a is a random variable not depending on j. Using the constraint that $\sum_{j=M}^{T-1} \beta_j^* = 1 - \beta_T$, we deduce

$$\sum_{j=M}^{T-1} \beta_j^* = \sum_{j=M}^{T-1} a(u_j + A\beta_j^*) = 1 - \beta_T,$$

which gives the solution

$$\beta_j^* = \frac{u_j + A\beta_j^*}{A + \sum_{i=M}^{T-1} u_i/(1 - \beta_T)} \tag{7}$$

since

$$\sum_{i=M}^{T-1} A\beta_i^* = A(1-\beta_T).$$

The above definition of β_j^* is equivalent to

$$\beta_j^* = \frac{u_j}{\sum_{i=M}^{T-1} u_i}(1-\beta_T), \tag{8}$$

which shows that the optimal β_j^* for $j = M, \ldots, T-1$ are unique when $M < T$ and $\beta_T < 1$. Moreover, we see from (7) and (8) that when there is no closing auction, which means $\beta_T = 0$, then the optimal benchmark is VWAP with $\beta_j^* = \frac{A\beta_j^* + u_j}{A + \sum_{i=M}^{T-1} u_i} = \frac{u_j}{\sum_{i=M}^{T-1} u_i}$. Note that β_j^* is $\mathcal{F}$-measurable provided that M is $\mathcal{F}$-measurable.

Step 2b: outer minimization problem over β_T and M

Thanks to (7), the optimization problem (6) becomes

$$\min_{\beta_T, M} \left\{ E\left[c|V| \max\left\{ \frac{1}{A + \sum_{i=M}^{T-1} u_i/(1-\beta_T)} \mathbb{1}_{\beta_T \neq 1}, \frac{1}{2A}\mathbb{1}_{\beta_T \neq 0} \right\} + Q(M) \right] \right\}. \tag{9}$$

We can write

$$\begin{aligned} &\max\left\{ \frac{1}{A + \sum_{i=M}^{T-1} u_i/(1-\beta_T)} \mathbb{1}_{\beta_T \neq 1}, \frac{1}{2A}\mathbb{1}_{\beta_T \neq 0} \right\} \\ &= \frac{1}{A} \max\left\{ \frac{1-\beta_T}{1-\beta_T + \sum_{i=M}^{T-1} u_i/A} \mathbb{1}_{\beta_T \neq 1}, \frac{1}{2}\mathbb{1}_{\beta_T \neq 0} \right\} \\ &= \frac{1}{A}\Bigg(\underbrace{\frac{1}{1 + \sum_{i=M}^{T-1} u_i/A}}_{\text{term 1}} \mathbb{1}_{\beta_T = 0} + \mathbb{1}_{\beta_T \neq 0} \underbrace{\max\left\{ \frac{1-\beta_T}{1-\beta_T + \sum_{i=M}^{T-1} u_i/A}, \frac{1}{2} \right\}}_{\text{term 2}} \Bigg). \end{aligned} \tag{10}$$

We analyze the following two cases separately:

1. If $\sum_{i=M}^{T-1} u_i > A$, then $\frac{1}{1+\sum_{i=M}^{T-1} u_i/A} < \frac{1}{2}$ so that term 1 is smaller than term 2 in (10). Therefore, it is optimal to then choose $\beta_T^* = 0$. Note that this choice is possible because $\sum_{i=M}^{T-1} u_i > A$ implies that $M < T$.

   In this case, the value of (10) minimized over β_T equals $\frac{1}{A+\sum_{i=M}^{T-1} u_i}$.

2. If $\sum_{i=M}^{T-1} u_i \leq A$, then $\frac{1}{1+\sum_{i=M}^{T-1} u_i/A} \geq \frac{1}{2}$ so that an optimal choice is $\beta_T^* \in (0, 1]$ with

$$\frac{1-\beta_T^*}{1-\beta_T^* + \sum_{i=M}^{T-1} u_i/A} \leq \frac{1}{2}. \tag{11}$$

Indeed, term 2 in (10) then equals 1/2 while term 1, which is equal to or greater than 1/2, is not relevant because $\beta_T^* > 0$.

In this case, the value of (10) minimized over β_T equals $\frac{1}{2A}$.

Combining these two cases, we obtain

$$\beta_T^* \begin{cases} = 0 & \text{if } \sum_{i=M}^{T-1} u_i > A, \\ \in (0,1] \text{ such that } \frac{1-\beta_T^*}{1-\beta_T^* + \sum_{i=M}^{T-1} u_i/A} \leq \frac{1}{2} & \text{if } \sum_{i=M}^{T-1} u_i \leq A, \end{cases} \tag{12}$$

and

$$\begin{aligned} &\min_{\beta_T} \frac{1}{A} \left(\frac{1}{1+\sum_{i=M}^{T-1} u_i/A} \mathbb{1}_{\beta_T = 0} + \mathbb{1}_{\beta_T \neq 0} \max \left\{ \frac{1-\beta_T}{1-\beta_T + \sum_{i=M}^{T-1} u_i/A}, \frac{1}{2} \right\} \right) \\ &= \frac{1}{A + \sum_{i=M}^{T-1} u_i} \mathbb{1}_{\sum_{i=M}^{T-1} u_i > A} + \frac{1}{2A} \mathbb{1}_{\sum_{i=M}^{T-1} u_i \leq A} \\ &= \frac{1}{A + \max\{\sum_{j=M}^{T-1} u_j, A\}}. \end{aligned}$$

Consequently, (9) is reduced to

$$\min_M E\left[\frac{c|V|}{A + \max\{\sum_{j=M}^{T-1} u_j, A\}} + Q(M) \right] = \min_M E\left[\frac{cE\left[|V||\mathcal{F}\right]}{A + \max\{\sum_{j=M}^{T-1} u_j, A\}} + Q(M) \right],$$

which is minimized by the minimizer of (2).

In the case of $\beta_T^* = 0$, it follows from (12) with $M = M^*$ that $\sum_{i=M^*}^{T-1} u_i > A$ and thus $M^* < T$. Hence, the optimal benchmark is VWAP with $\beta_j^* = \frac{u_j}{\sum_{i=M}^{T-1} u_i}$, as proven in Step 2a.

If $\beta_T^* > 0$, we saw above in the second case that the value of (10) minimized over β_T equals $\frac{1}{2A}$ so that

$$\frac{cE\left[|V||\mathcal{F}\right]}{A + \max\{\sum_{j=M}^{T-1} u_j, A\}} + Q(M) = \frac{cE\left[|V||\mathcal{F}\right]}{2A} + Q(M)$$

on $\beta_T^* > 0$. Because Q is strictly decreasing, we obtain the optimal $M^* = T$, which then implies that $\beta_T^* = 1$. Note that condition (11) is satisfied for this choice. Therefore, the only optimal $\beta_T^* > 0$ is $\beta_T^* = 1$, and we obtain the dichotomy presented in the main result: either all weight is in the auction or the benchmark is VWAP without an auction.